International Journal of Theoretical Physics, Vol. 17, No. 6 (1978), pp. 389–504

Space–Time Edge Geometry

C. T. J. Dodson

Department of Mathematics, University of Lancaster

Received July 5, 1978

CONTENTS

0020–07748/78/0600–0389$05.00/0

Georgia Institute of Technology

Volume 17 Number 6 June 1978
Space–Time Edge Geometry, *C. T. J. Dodson* 389

Preface

My objective in this monograph is to provide a mathematically rigorous account of the handling of geometrical singularities in space–time manifolds that will be sufficient to bring newcomers to the field into touch with current research. Some new material and new proofs are included. Throughout I have made an effort to bring results and their proofs, new or from the literature, into a rigorous common accord and a natural juxtaposition. By this means a clear theme of development is apparent, but the standardization of notation and argument permit the text to be used as a reference work because of the detailed labeling of subsections and the large number of cross-references.

The readers will, I hope, include nonspecialists as well as those already having an interest in space–time theory. To differential or global geometers we can offer a way of using fibre bundles to study manifold singularities that develops to a definite and exciting physical application; it is well within their reach because the required background physics is very slight and amply covered in the text. The first part is directed to astronomers and physicists without some knowledge of modern differential geometry but who wish to grasp the current definition and classification of space–time singularities. This part reviews necessary material on topology, manifolds, Lie groups, fibre bundles, and connections; it omits all proofs but substitutes worked examples to illustrate most of the definitions. So Part I can either be scanned for unfamiliar topics or referred to for definitions and examples during a study of the sequel. Since all but elementary definitions are included, this paper is self-contained.

The first half is concerned with the geometry of manifolds with a connection and of the fibre bundles related to them; specialization to space–time manifolds is deferred to Part III. A connection on a manifold is equivalent to having well-defined covariant differentiation or parallel transport. Such extra structure is possessed by a space–time manifold by virtue of its pseudo-Riemannian metric tensor field, which always determines the unique Levi–Cività connection. Our story begins in 1971, when B. G. Schmidt used this connection to effect a completion that incorporates singularities as extra points in an extended space–time. His process is efficient and attractive: the frame bundle is metrized, Cauchy completed, and then factored by its structure group, whose action extends uniquely to the completion. We accumulate general properties of the spaces involved and subsequently concentrate on their role in space–time theory. The uniqueness is of course advantageous in defining singularities, a notoriously difficult thing in space–times. Their derivation from the connection assures us of physical significance, for in cosmological models it is the disposition of matter that determines the curvature of the connection. Moreover, it turns out that the boundary points in the completed space–time are typically endpoints of curves on which the curvature is without limit. However, drawbacks arise because the completion need only be a topological space of non-Hausdorff type: we lose the manifold structure and usual separation properties for events near the edge. So we consider modifications that remove some of the unphysical identifications among singularities. One of these modifications, due to C. J. S. Clarke, appears to meet most of the immediate requirements and provides our tale with an unexpected twist. To firmly grasp just what goes wrong with Schmidt's original completion, we make a detailed study of a simple class of space–times that yield unwanted identifications of singularities. We end with the currently accepted classification scheme for singularities and summarize the position as it appears after the Eighth International Conference on General Relativity and Gravitation (GR8), held in August 1977.

Most of the work on this study was done during a sabbatical year 1976–1977 at the International Center for Theoretical Physics, Trieste, Italy, where a course of lectures on the material was given. I wish to thank the director, Abdus Salam, for his encouragement and interest, and his staff in the ICTP Library and Publications Office for their invaluable support. My stay in Trieste was made possible by the award of a Royal Society European Science Exchange Programme Fellowship. I am indebted to the Royal Society for this and for a travel grant to attend the GR8 Conference in Canada, where I was kindly supported by the Society for General Relativity and Gravitation. I am grateful to Professor Salam, the IAEA, and UNESCO for hospitality in Trieste and for a travel grant to make the visit. I have consulted too many people for due acknowledgment to be given here; in fact many of the workers mentioned in the text have kindly responded to queries with letters and preprints. However, in particular, C. J. S. Clarke has been unstintingly helpful throughout, and M. J. Slupinski helped revise the final draft and collaborated on the preparation of Section III.4.4. Finally, many people have been subjected to lectures on the material (at ICTP and Lancaster) or to seminars on space–time completions (in Europe and North America) during the last year or so, and their comments and queries have led to improvements in this text. I am pleased to be able to acknowledge these sources of help in organizing the contents, but I remain fully responsible for any remaining errors.

Part I. Background with Examples

The material that it is convenient to have available for reference in the sequel we gather under five section headings: (1) topology, (2) manifolds, (3) Lie groups, (4) fibre bundles, and (5) connections.

For topology, which pertains mainly to separation and convergence properties, conveniently readable and widespread references are Kelley [39] and Dieudonné [14]. Little in the way of amplification seems necessary for that section.

Differential geometry is covered, though rather tersely for physicists, in Kobayashi and Nomizu [41]. The texts by Bishop and Crittenden [4] and Brickell and Clark [8] deal with similar material but contain more breathing spaces in the exposition and more actively encourage an intuitive feel for the subject. A geometrically motivated account of manifolds, their curvature, and connections is given in Dodson and Poston [16], which begins with vector spaces and proceeds to a rigorous formulation of relativistic space–time. Our notation follows this text when it is convenient; otherwise it is similar to that in the former books. The end or absence of a proof is signified by ■.

1. TOPOLOGY

By the term *space* we always mean a topological space, perhaps also having additional structure that is plain from the context: A map between spaces, denoted by f say, will be represented as

$$f\colon X \to Y\colon x \mapsto f(x)$$

when f has domain X and it sends the arbitrary element $x \in X$ to $f(x) \in Y$. Moreover, we shall use the notation

$$f^{\leftarrow}\colon \text{sub } Y \to \text{sub } X\colon B \mapsto \{x \in X \mid f(x) \in B\}$$

for the induced map on subsets of Y when the image of f is Y. Thus we reserve the symbol f^{-1} for the case when f is a bijection and f^{-1} is its inverse. The identity map on any space X is denoted I_X.

1.1. A space is called

(i) T_0 if for each pair of distinct elements from it there is a neighborhood of one to which the other does not belong;

(ii) T_1 if each set consisting of a single point is closed;

(iii) T_2 or *Hausdorff* if whenever x and y are distinct elements from it there exist disjoint neighborhoods of x and y. Hence metric spaces are always T_2 spaces.

1.2. Every subspace of a T_1 [respectively, T_2] space is a T_1 [respectively, T_2] space. ■

1.3. For any semimetric space (X, ρ) we have the equivalences X is $T_2 \Leftrightarrow X$ is $T_1 \Leftrightarrow \rho$ is a metric. ■

1.4. Every convergent sequence in a T_2 space has a unique limit. ■

1.5. Let $g, f\colon X \to Y$ be continuous maps with Y a T_2 space. Then $\{x \in X \mid f(x) = g(x)\}$ is closed; if it is also dense in X, then $f = g$. ■

1.6. Let $f\colon X \to Y$ be continuous with X a T_2 space. Then graph $f = \{(x, f(x)) \in X \times Y\}$ is closed in the product topology and $f^{\leftarrow}(y)$ is closed in X for all $y \in Y$. ■

1.7. A space is called

(i) *regular* if for any subset A and point $x \in A$ there exist disjoint open sets containing A and x;
(ii) T_3 if regular and T_1;
(iii) *normal* if given any two disjoint closed subsets there are disjoint open subsets containing them;
(iv) T_4 if normal and T_1.

1.8. Every semimetric space is regular and normal but not necessarily Hausdorff. ■

1.9. A map $f\colon X \to Y$ between metric spaces (X, d) and (Y, d') is called *uniformly continuous* if $(\forall \epsilon > 0)(\exists \delta > 0)$:

$$d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \epsilon$$

1.10. Let A be a dense subset of a metric space (X, d) and f a uniformly continuous map of A into a complete metric space (Y, d'). Then there exists a uniformly continuous map $\bar{f}\colon X \to Y$, coinciding with f in A. ■

1.11. Let (X, d), (X, d') be metric spaces having the same underlying space. Then we have the following:

(i) If I_X is a homeomorphism, then the d-topology coincides with the d'-topology for X.
(ii) If I_X is uniformly continuous (with a uniformly continuous inverse) between (X, d) and (X, d'), then the Cauchy sequences for d and d' agree. ■

In case (i) we call d and d' *topologically equivalent* distances, and in case (ii) we call them *uniformly equivalent*; from 1.9 the latter implies the former.

1.12. Let A be a subset of a topological space X. We shall use the following notation:

(i) int (A) for the *interior* of A, the largest open subset of A;
(ii) A^c for the *closure* of A, the smallest closed set containing A;
(iii) $\dot{A}$ for the *topological boundary* of A, the set $A^c \cap (X\backslash A)^c$.

We recall a few properties.

(a) A is open $\Leftrightarrow \text{int}(A) = A \Leftrightarrow A \cap \dot{A} = \varnothing$.
(b) A is closed $\Leftrightarrow \dot{A} \subseteq A \Leftrightarrow A$ contains its limit points.
(c) $(X\backslash A)^c = X\backslash\text{int}(A)$.
(d) $A^c\backslash\text{int}(A) = \dot{A}$.
(e) $A \cup \dot{A} = A^c$.
(f) $A\backslash\dot{A} = \text{int}(A)$. ■

1.13. A *covering space* of a connected, locally arcwise connected topological space X, is a connected space E such that

(i) there exists a *projection* $p: E \to X$;
(ii) for all $x \in X$ there exists a connected open neighborhood U of x such that each connected component $p^{\leftarrow}(U)'$ of $p^{\leftarrow}(U)$ is open in E and homeomorphic to U by the restriction of $p: p^{\leftarrow}(U)' \simeq U$.

A covering space is called a *universal covering space* if it is simply connected.

If X is a manifold, then every covering space has a unique manifold structure that makes the projection differentiable (cf. 4.6). ■

Two covering spaces $p: E \to X$, $p': E' \to X$ are *isomorphic* if there exists a homeomorphism $f: E \simeq E'$ such that $p' \circ f = p$.

Note that we use $\simeq$ to denote that two spaces are homeomorphic, that is, topologically equivalent. We shall use $\cong$ to denote a stronger equivalence that is of interest when additional structure is present in the spaces. For example, in the case of metric spaces (X, d), (Y, d'), the existence of a homeomorphism $(X \simeq Y)$ that is also an isometry will be indicated by $X \cong Y$. Similarly in the case of manifolds M, M' (cf. Section 2) if their underlying topological spaces are homeomorphic $(M \simeq M')$ via a map f that is also a *diffeomorphism* (f and f^{-1} are continuously differentiable), then we write $M \cong M'$.

1.14. Paracompactness. An *open covering* of a topological space X is a family F of open sets W such that

$$X = \bigcup_{W \in F} W$$

This family is called *locally finite* if for all $x \in X$ there is an open set W_x containing x and such that

$$\{W \in F \mid W \cap W_x \neq \varnothing\}$$

is finite. The space X is *paracompact* if every open covering F of X has a locally finite *open refinement* $H \subseteq F$; that means that H is to be an open covering of X with

$$(\forall V \in H)(\exists W \in F) \colon V \subset W$$

It is known that all metric spaces are paracompact. ■

2. MANIFOLDS

By the term *manifold* we shall mean a finite-dimensional, smooth real manifold. For the standard manifolds like $\mathbb{R}$, $\mathbb{R}^n$, and S^1 and their products we shall assume atlases derived from the identity map on the relevant $\mathbb{R}^m$ that contains them. We recall that every point x in a *topological* n-manifold M has a neighborhood *homeomorphic* to an open set in $\mathbb{R}^n$. For a *smooth* n-manifold M, each such x also possesses a tangent vector space T_xM; these tangent spaces can be collected to form a smooth $2n$-manifold, the *tangent bundle* TM, with underlying set $\bigcup_{x\in M} T_xM$ and *canonical projection* $\Pi_T \colon TM \to M$, sending T_xM to x for all $x \in M$. Each of the tangent spaces T_xM is isomorphic to $\mathbb{R}^n$, and so we shall often denote an element in T_xM by a pair (x, v) with v in some convenient isomorph of $\mathbb{R}^n$, induced by a choice of basis for T_xM.

The differentiable structure of a smooth manifold allows us to define differentiability for maps between manifolds. In particular every point of such an n-manifold has a neighborhood that is *diffeomorphic* (denoted by $\cong$) to an open set in $\mathbb{R}^n$. Suppose that M_1 and M_2 are smooth manifolds of dimensions n and m, respectively. Given any differentiable map

$$f \colon M_1 \to M_2$$

there is induced a map, its *differential*,

$$Df \colon TM_1 \to TM_2$$

which for all $x \in M_1$ has a linear restriction

$$D_xf \colon T_xM_1 \to T_{f(x)}M_2$$

If φ and ψ are local chart maps about x and $f(x)$, then we have a local representation of f (with domain *contained in* $\mathbb{R}^n$ signified by $\rightarrowtail$)

$$\psi \circ f \circ \varphi^{-1} \colon \mathbb{R}^n \rightarrowtail \mathbb{R}^m \colon (x^i) \mapsto (y^j)$$

The Jacobian matrix $(f_i^j)_x$ at $(x^i) = \varphi(x)$ induces the local representation of the derivative D_xf by[1]

$$\tilde{D}_xf \colon \mathbb{R}^n \to \mathbb{R}^m \colon (v^i) \mapsto (f_i^j v^i)$$

[1] We use Einstein's summation convention throughout.

We shall find it convenient to write

$$D_x f\colon T_x M_1 \to T_{f(x)} M_2\colon (x, v^i) \mapsto (f(x), f_i^j v^i)$$

when a choice of charts has been decided.

2.1. A *vector field* on a manifold M is a map

$$w\colon M \to TM$$

such that for all $x \in M$, $\Pi_T(w(x)) = x$. (See 4.3, Example 3.) We shall always assume that our vector fields are *smooth*. An equivalent definition is that w is a smooth *section* of the surjection Π_T. The set of such will be denoted $\mathcal{T}M$; under pointwise operations it becomes a vector space (infinite dimensional).

It turns out that vector fields are *derivations* on smooth real functions on the manifold.

2.2. The *Lie bracket* of two vector fields v, w on a manifold M is the unique vector field denoted $[v, w]$ and defined to act on all smooth $f\colon M \to \mathbb{R}$ by

$$[v, w](f) = v(w(f)) - w(v(f)) \blacksquare$$

2.3. An *integral curve* of a vector field w on M is a curve $c\colon t \mapsto c(t)$ in M such that its tangent vector $\dot{c}(t)$ at each t satisfies $\dot{c}(t) = w \circ c(t)$. Such curves always exist for smooth vector fields and they are essentially unique (see [16] for a recent geometrical proof). ■

2.4. Suppose that (M, g) is a connected Riemannian manifold. Then g defines a *distance function*

$$d_g\colon M \times M \to \mathbb{R}\colon (x, y) \mapsto \inf\left\{\int \|\dot{c}\| \,\middle|\, c \in \Gamma(x, y)\right\}$$

where $\Gamma(x, y)$ is the class of all piecewise continuously differentiable curves from x to y in M. Thus we have

(i) (M, d_{g}) is a topological metric space, and the metric topology coincides with the manifold topology.

(ii) (M, g) is *complete* if (M, d_{g}) is a complete metric space. A Riemannian manifold that is not complete admits a completion as a metric space by the standard procedure using equivalence classes of Cauchy sequences. We shall denote the extension of a metric d to this completion by $\bar{d}$. ■

Example. The point we illuminate is that if (M, g) is not complete, then for some $x, y \in M$ there may be no $c \in \Gamma(x, y)$ such that the length $\int \|\dot{c}\|$ of c equals the distance $d_{\mathrm{g}}(x, y)$. Take $M = \mathbb{R}^2 \backslash \{(0, 0\}$ with the Euclidean metric and choose $x = (-1, 0)$, $y = (1, 0)$. Then plainly $d_g(x, y) = 2$, but there is no curve of this length joining x and y in M. In this case the *Cauchy completion* $\bar{M}$ of M, is easily seen to be $\mathbb{R}^2$.

2.5. A connected Riemannian manifold turns out to be complete if and only if *every geodesic can be extended to arbitrarily large parameter values*; however, this correctly belongs in Section 5 under *connections* (see 5.8, 5.11, and 5.12). ■

2.6. Every *paracompact* manifold admits a Riemannian structure. Conversely, a connected Riemannian manifold is also a metric space by 2.4 and hence paracompact by 1.14. For a connected manifold M, paracompactness is equivalent to M satisfying the second axiom of countability: its topology can be generated by a countable collection of open sets (see Kobayashi and Nomizu [41], p. 271). ■

2.7. A *parallelization* on an n-manifold is a continuous assignation of an ordered set of n independent tangent vector fields. That is a continuous assignation of a *frame* or equivalently a continuous *section* of the *frame bundle* (see 4.1). A manifold is called *parallelizable* if it admits a parallelization. The product manifold of parallelizable manifolds is parallelizable. The only compact two-manifolds that are parallelizable are the Klein bottle and torus, and the only parallelizable spheres are S^1, S^3, and S^7. Any manifold with a global chart is parallelizable. If an n-manifold M is parallelizable, then there is a diffeomorphism $TM \cong M \times \mathbb{R}^n$. ■

3. LIE GROUPS

Global algebraic actions on manifolds prove to be powerful investigative tools, and they provide elegant geometrical constructions. We are interested in the actions of groups on smooth manifolds, so we shall want the groups to be smooth themselves with smooth actions. We are forced to the following definition.

3.1. A *Lie group* is a group that is also a smooth manifold such that the group operation $(a, b) \mapsto ab^{-1}$ is smooth.

Note that saying that $(a, b) \mapsto ab^{-1}$ is smooth is simply a short way of saying that the following maps are smooth for all a, b in the group:

$$\begin{array}{ll} L_a : b \mapsto ab & \text{(left translation)} \\ R_a : b \mapsto ba & \text{(right translation)} \\ {}^{-1} : a \mapsto a^{-1} & \text{(inversion)} \end{array}$$

Example 1. The *general linear group* $Gl(n; \mathbb{R})$ of all $n \times n$ real nonsingular matrices forms an open submanifold of $\mathbb{R}^{n^2}$. In particular $Gl(1; \mathbb{R}) = \mathbb{R}\backslash\{0\}$, which we shall denote by $\mathbb{R}^*$.

Example 2. The group of matrices

$$\begin{bmatrix} \dfrac{1}{(1-v^2)^{1/2}} & \dfrac{v}{(1-v^2)^{1/2}} \\ \dfrac{v}{(1-v^2)^{1/2}} & \dfrac{1}{(1-v^2)^{1/2}} \end{bmatrix} \quad \text{with } 1 > v^2 \in \mathbb{R}$$

This is important in relativity where it generates a one-parameter subgroup of the *Lorentz group*. It has an equivalent representation as

$$\begin{bmatrix} \cosh \chi & \sinh \chi \\ \sinh \chi & \cosh \chi \end{bmatrix} \quad \text{with } \chi \in \mathbb{R}$$

3.2. A *left-invariant vector field* of a Lie group G is a vector field $w\colon G \to TG$ that is fixed under the differentials of left translations. This means that for all $g \in G$, $L_g\colon G \to G\colon h \mapsto gh$, left translation by g, and the differential $DL_g\colon TG \to TG$ have the property that

$$DL_g w(h) = w(L_g(h)) = w(gh), \qquad \forall h \in G$$

Example. Let $G = Gl(n; \mathbb{R})$. Then for any $a \in G$,

$$T_aG = \{(a, A) | A \text{ is an } n \times n \text{ real matrix}\}$$

In particular for $n = 1$ we have $G = \mathbb{R}^*$, and for all $g \in \mathbb{R}^*$

$$L_g\colon \mathbb{R}^* \to \mathbb{R}^*\colon a \mapsto ga$$

$$D_aL_g\colon T_a\mathbb{R}^* \to T_{ga}\mathbb{R}^*\colon (a, A) \mapsto (ga, gA)$$

3.3. The left-invariant vector fields of a Lie group G form a vector space, and an algebra under the Lie bracket composition called the *Lie algebra* $\mathfrak{g}$ of G.

As a vector space we have an isomorphism of $\mathfrak{g}$ with the tangent space to the identity $e \in G$

$$\mathfrak{g} \to T_eG\colon \gamma \mapsto \gamma(e)$$

Hence $\mathfrak{g}$ has the same dimension as the manifold G.

As an algebra we find that the inclusion of $\mathfrak{g}$ in $\mathfrak{X}(G)$, the Lie algebra of *all* vector fields on G, is a homomorphism so $\mathfrak{g}$ is a *Lie subalgebra* of $\mathfrak{X}(G)$. ■

Example. For $G = Gl(n; \mathbb{R})$ we have $\mathfrak{g} = \mathfrak{gl}(n; \mathbb{R})$ consisting essentially of all $n \times n$ real matrices A, which determine fields by

$$A\colon G \to TG\colon a \mapsto (a, A) \in T_aG$$

The Lie bracket operation is the composite matrix product

$$[A, B] = AB - BA$$

We pursue the one-dimensional example in 3.2 to display the left-invariance property. Plainly we have $\mathfrak{gl}(1; \mathbb{R}) = \mathbb{R}$. Let $\gamma \in \mathbb{R}$; then we require

$$\gamma: \mathbb{R}^* \to T\mathbb{R}^*: a \mapsto (a, \gamma_a)$$

such that for all $a, g \in \mathbb{R}^*$

$$DL_g\gamma(a) = \gamma(L_g(a))$$

Hence at the identity $e = 1 \in \mathbb{R}^*$

$$g\gamma(e) = \gamma(ge) = \gamma(g)$$

So we have $\gamma(a) = a\gamma(1) = a\gamma_1$, say.

3.4. The *adjoint representation* Ad of a Lie group G in its Lie algebra $\mathfrak{g}$ is obtained from the automorphisms

$$\text{ad}\,(g) = L_g R_{g^{-1}}: G \to G: h \mapsto ghg^{-1}, \qquad \forall g \in G$$

by putting

$$\text{Ad}\,(g): \mathfrak{g} \to \mathfrak{g}: \gamma \mapsto D(L_g R_{g^{-1}})\gamma = D\,\text{ad}\,(g)(\gamma)$$

This of course implies that left-invariance is preserved ■

Example. The adjoint representation is trivial for any $g \in \mathbb{R}^*$, since

$$L_g R_{g^{-1}}: \mathbb{R}^* \to \mathbb{R}^*: h \mapsto ghg^{-1} = h$$

so $\text{Ad}\,(g) = I_{\mathbb{R}}$.

3.5. Every γ in $\mathfrak{g}$, the Lie algebra of a Lie group G, generates a *one-parameter subgroup* of G as follows.

Let $c_\gamma: t \mapsto \gamma_t$ be the integral curve in G (see 2.3), determined for $|t| < \epsilon$ for some real $\epsilon > 0$, with initial conditions: $\gamma_0 = c_\gamma(0) = e$, the identity in G, and $\dot{c}_\gamma(0) = \gamma(e) \in T_eG$ (see 3.3). Define the map

$$\varphi_t: G \to G: g \mapsto L_g(\gamma_t), \qquad |t| < \epsilon$$

Now this is valid for all $g \in G$ and hence admits an extension to all $t \in \mathbb{R}$. It turns out that we have a group, because for all $g \in G$

$$L_g(\gamma_{t+s}) = L_g(\gamma_t) \circ L_g(\gamma_s)$$

Hence the required subgroup of G is

$$G_\gamma = \{\varphi_t(e) \mid t \in \mathbb{R}\}$$

We also obtain *the exponential map*

$$\exp: \mathfrak{g} \to G: \gamma \mapsto \gamma_1$$

Example. For $G = Gl(n: \mathbb{R})$ we find that the exponential map coincides with the usual exponential function for matrices γ

$$\exp \gamma = \sum_{k=0}^{\infty} \gamma^k/k!$$

We can see why this is so from the case $n = 1$. We seek the integral curve for $\gamma \in \mathfrak{gl}(1; \mathbb{R}) = \mathbb{R}$ given by (see Example, 3.3)

$$c_\gamma: (-\epsilon, \epsilon) \to \mathbb{R}^*$$

subject to $c_\gamma(0) = e = 1$ and $\dot{c}_\gamma(0) = \gamma(1) = \gamma_0$ at $0 \in (-\epsilon, \epsilon)$ and subject at all $t \in (-\epsilon, \epsilon)$ to

$$\dot{c}_\gamma(t) = \gamma \circ c_\gamma(t)$$

So our differential equation is

$$\dot{c}_\gamma(t) = \gamma_0 c_\gamma(t)$$

and the required solution is

$$c_\gamma: (-\epsilon, \epsilon) \to \mathbb{R}^*: t \mapsto e^{\gamma_0 t}$$

Finally, the one-parameter subgroup is given by

$$\varphi_t: \mathbb{R}^* \to \mathbb{R}^*: g \mapsto g e^{\gamma_0 t}$$

and the exponential map is

$$\exp: \mathbb{R} \to \mathbb{R}^*: \gamma \mapsto e^\gamma$$

3.6. Let G be a Lie group and P a smooth manifold. Then G *acts on P to (or on) the right* if there is a smooth map

$$P \times G \to P: (u, g) \mapsto R_g(u)$$

satisfying (i) $g: P \to P: u \mapsto R_g(u)$ is a diffeomorphism, $\forall g \in G$, and (ii) $R_{gh}(u) = R_h(R_g(u))$, $\forall g, h \in G$, $\forall u \in P$.

Example 1. Take $G = \mathbb{R}^*$ and $P = \mathbb{R}^2$ with

$$\mathbb{R}^2 \times \mathbb{R}^* \to R^2: ((x, y), g) \mapsto (gx, gy)$$

Example 2. Take $P = G$ and use right translation in G:

$$G \times G \to G: (h, g) \mapsto R_g(h) = hg$$

3.7. Let a Lie group G act on the right of a manifold P. Then, from 3.5, every $\gamma \in \mathfrak{g}$ determines a one-parameter subgroup of G

$$G_\gamma = \{\varphi_t(e) = \exp t\gamma \mid t \in \mathbb{R}\}$$

which acts on the right of P by its inclusion in G; it also has the following properties:

(i) Through each $u \in P$ there is a smooth curve

$$t \mapsto R_{\varphi_t(e)}(u)$$

with tangent vector $\gamma_u{}^* \in T_u P$ at $t = 0$ (see 4.2).

(ii) Denote by $\mathfrak{X}P$ the Lie algebra of vector fields on P, and define a map

$$\Phi: \mathfrak{g} \to \mathfrak{X}P: \gamma \mapsto \gamma^*$$

where

$$\gamma^*: P \to TP: u \mapsto \gamma_u{}^*$$

Then Φ is a Lie algebra homomorphism. If the only element of G with a fixed point is e, then $\gamma_u{}^*$ is not the zero vector at any $u \in P$, for nonzero $\gamma \in \mathfrak{g}$. ■

Example 1. When $G = O(2; \mathbb{R})$, the orthogonal subgroup of $Gl(2; \mathbb{R})$, then for any

$$\gamma = \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix} \in \mathfrak{g}$$

we have

$$\exp \gamma = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Example 2. For $G = Gl(1; \mathbb{R}) = \mathbb{R}^*$ we have from 3.5, for any $\gamma \in \mathfrak{g}l(1; \mathbb{R}) = \mathbb{R}$,

$$G_\gamma = \{e^{t\gamma} = \exp t\gamma \mid t \in \mathbb{R}\} \subset \mathbb{R}^*$$

Using the action of $\mathbb{R}^*$ on $\mathbb{R}^2$ from Example 1, 3.6, we find the curve

$$t \mapsto R_{e^{t\gamma}}(u) = (xe^{t\gamma}, ye^{t\gamma}) = u(t)$$

in $\mathbb{R}^2$ through $u = (x, y)$. The tangent vector to this curve at any t is

$$((xe^{t\gamma}, ye^{t\gamma}), (\gamma xe^{t\gamma}, \gamma ye^{t\gamma})) \in T_{u(t)}\mathbb{R}^2$$

Evaluation at $t = 0$ gives the map

$$\gamma^*: \mathbb{R}^2 \to T\mathbb{R}^2: u \mapsto (u, \gamma u)$$

Evidently for this example $\gamma_u{}^*$ is the zero vector at $u = (0, 0)$ because all $\gamma \in \mathbb{R}$ have a fixed point there.

3.8. An action of a Lie group G on the right of a manifold P is called

(i) *transitive* if $(\forall u, v \in P)\exists g \in G: R_g(u) = v$;

(ii) *free* if the only element of G with a fixed point is e;

(iii) *effective* if $(R_g(u) = u, \forall u \in P) \Rightarrow g = e$.

Example 1. The action of $\mathbb{R}^*$ on $\mathbb{R}^2$ in 3.6 is not transitive, but the right translation in any G is always transitive because

$$u, v \in G \Rightarrow (\exists g = u^{-1}v): R_g(u) = v$$

Example 2. Let $P = S^1 \times \mathbb{R}^*$; then the action induced by right translation in $\mathbb{R}^*$ is free, but the aforementioned action of $\mathbb{R}^*$ on $\mathbb{R}^2$ is not free because every element of $\mathbb{R}^*$ has a fixed point at the origin.

3.9. Given a Lie group G acting on the right of a manifold P, the *orbit of G through $u \in P$* is the set

$$[u] = \{R_g(u) | g \in G\}$$

and we denote by P/G the set of all such $[u]$ for $u \in P$. Evidently P/G gains a topology by requiring that the projection

$$\Pi_P: P \rightarrow P/G: u \mapsto [u]$$

be continuous.

We shall denote the *connected component* of the identity in a Lie group G by G^+. The notation is motivated by the special case

$$Gl(1; \mathbb{R}) = \mathbb{R}^* \quad \text{and} \quad \mathbb{R}^+ = Gl(1; \mathbb{R})^+ = \{g \in \mathbb{R} | g > 0\}$$

Remark. A detailed study of the geometry of the Lie group of unimodular operators on $\mathbb{R}^2$, $SL(2; \mathbb{R})$, with many pictures, is given in [16] as an example of an abstract space with a natural pseudo-Riemannian structure.

4. FIBRE BUNDLES

The general linear group and its subgroups always have actions on tangent vectors on a manifold. The tangent vectors collectively form a manifold, the tangent bundle, so we have a ready-made action on this manifold. However, at any point in a tangent bundle any element of the general linear group simply sends the point to another one belonging to the same tangent space, in a linear fashion. So the orbits of the general linear group in this action are vector spaces. For wider application we shall be interested in actions of Lie groups that give rise to orbits that are manifolds and not necessarily vector spaces. This is the motivation for defining fibre bundles.

4.1. A *principal fibre bundle* over a manifold M is a manifold P with a Lie group G such that

(i) G acts freely on P to the right;

(ii) $M = P/G$ and the canonical projection $\Pi_P: P \to M$ is smooth;

(iii) P is *locally trivial*; that is, every $x \in M$ has a neighborhood U such that $\Pi_P^{\leftarrow}(U)$ is diffeomorphic to $U \times G$.

We call G the *structure group* of the bundle; property (ii) ensures that it is transitive on all *fibres* $\Pi_P^{\leftarrow}(x)$. We shall refer to the principal fibre bundle as (P, G, M), or simply as P, when G and M are well defined by the context.

Example 1. The trivial product bundle with $P = M \times G$.

Example 2. The *frame bundle* or *bundle of linear frames* over a smooth manifold M. Here we have

$$P = LM = \{(x, (X_i)) | x \in M; (X_i) \text{ an ordered basis for } T_xM\}$$

$G = Gl(n; \mathbb{R})$, where n is the dimension of M. For all $x \in M$ any choice of basis for T_xM induces an isomorphism $T_xM \cong \mathbb{R}^n$ which allows $Gl(n; \mathbb{R})$ to act by matrix multiplication on the coordinate vector. Hence the subset determined by $(x, (X_i)) \in LM$, that is, the orbit through this element (cf. 3.9), is

$$[(x, (X_i))] = \{(x, (X_j g_i^j)) | (g_i^j) = g \in Gl(n; \mathbb{R})\}$$

Now, as g runs through $Gl(n; \mathbb{R})$, so $R_g(x, (X_i))$ runs through all bases for T_xM; therefore we may as well abbreviate $[(x, (X_i))]$ to x. This leaves the required projection map in the form

$$\Pi_L: LM \to M: (x, (X_i)) \mapsto x$$

Example 3. We shall later have occasion to use LS^1. Here $M = S^1$ and $G = \mathbb{R}^*$ and so $LS^1 \cong S^1 \times \mathbb{R}^*$, the trivial product bundle. It has two components, corresponding to positively oriented bases $L^+S^1 \cong S^1 \times R^+$ and negatively oriented bases $L^-S^1 \cong S^1 \times R^-$. Quite generally, if M is an *orientable* manifold then LM has two components; otherwise LM is connected. Any parallelizable manifold is orientable (see 2.7).

4.2. Let (P, G, M) be a principal fibre bundle. For all $\gamma \in \mathfrak{g}$ the fundamental vector field corresponding to γ is γ^*, as defined in 3.7. We observe the following properties.

(i) Since the action of G translates each fibre along itself, γ_u^* is tangent to the fibre at all $u \in P$. Also G acts freely on P so $\gamma_u^* \neq 0 \in T_uP$ unless $\gamma = 0 \in \mathfrak{g}$. In consequence the map

$$\mathfrak{g} \to T_uP: \gamma \mapsto \gamma_u^*$$

has trivial kernel, and since it is linear, it is an injection for all $u \in P$ (see 4.4).

(ii) Another formulation of γ^* is sometimes useful. For all $u \in P$ define a map

$$\sigma_u: G \to P: g \mapsto R_g(u)$$

This induces a bundle map, $D\sigma_u: TG \to TP$, which at the identity $e \in G$ has the restriction

$$T_eG \to T_uP: \gamma \mapsto \gamma_u^*$$

where we have used the isomorphism in 3.3 to identify the spaces T_eG and $\mathfrak{g}$.

(iii) For all $\gamma \in \mathfrak{g}$ and all $u \in P$ with $\Pi_P(u) = x$,

$$D\Pi_P(\gamma_u^*) = 0 \in T_xM$$

This is another expression of the tangency of γ_u^* to the fibre containing u, mentioned in (i).

Example 1. Consider the case of a frame bundle LM where $G = Gl(n; \mathbb{R})$ and $\mathfrak{g} = \mathfrak{g}l(n; \mathbb{R})$. Then we can display the preceding maps in components as follows. Let $u = (x^i, b_j^i) \in LM$.

$$\sigma_u: G \to LM: (g_j^i) \mapsto (x^i, g_m^i b_j^m)$$

$$D\sigma_u: T_eG \to T_uLM: (\delta_j^i, \gamma_j^i) \mapsto (x^i, b_j^i, 0, \gamma_m^i b_j^m)$$

$$\mathfrak{g} \to T_uLM: (\gamma_j^i) \mapsto (\gamma_m^i b_j^m)$$

$$(\gamma_j^i)^*: LM \to TLM: u \mapsto (x^i, b_j^i, 0, \gamma_m^i b_j^m)$$

Example 2. Consider the case of $LS^1 = S^1 \times \mathbb{R}^*$. We have the following maps, for all $\gamma \in \mathfrak{g}l(1, \mathbb{R})$ and $u = (x, b) \in LS^1$.

$$\gamma^*: LS^1 \to TLS^1: (x, b) \mapsto (x, b, 0, \gamma b)$$

$$\sigma_u: \mathbb{R}^* \to LS^1: g \mapsto (x, gb)$$

$$D\sigma_u: T_1\mathbb{R}^* \to T_uLS^1: (1, \gamma) \mapsto (x, b, 0, \gamma b), \quad \text{at } g = 1$$

$$T_1\mathbb{R}^* \cong \mathbb{R}: (1, \gamma) \mapsto \gamma$$

4.3. Let (P, G, M) be a principal fibre bundle, and let F be a manifold on which G acts on the left. Then the *fibre bundle associated to* (P, G, M) *with fibre* F is a manifold $(P \times F)/G$ defined by the following properties:

(i) The right action of G on $P \times F$ is

$$(P \times F) \times G \to P \times F: (u, a, g) \mapsto (R_g(u), L_{g^{-1}}(a))$$

(ii) The projection map is

$$\tilde{\Pi}_P: (P \times F)/G \to M: R_G(u, a) \mapsto \Pi_P(u)$$

and it is required to be smooth; also every $x \in M$ has a neighborhood U such that $\tilde{\Pi}_P{}^{\leftarrow}(U)$ is diffeomorphic to $U \times F$.

Example 1. Take $F = G$ and use left translation.

Example 2. The tangent bundle TM to a manifold M can be considered the bundle associated to the frame bundle LM with fibre $\mathbb{R}^n$, when M has dimension n. The fibre $\Pi_L{}^{\leftarrow}(x)$ over each $x \in M$ can be identified with the tangent space T_xM to M at x. In particular we see this for S^1 with $F = \mathbb{R}^1 \cong \mathbb{R}$, $G = \mathbb{R}^*$. We have

$$(LS^1 \times \mathbb{R}) \times \mathbb{R}^* \to LS^1 \times \mathbb{R}: (x, b, a, g) \mapsto (x, bg, g^{-1}a)$$

for the right action of $\mathbb{R}^*$ on $LS^1 \times \mathbb{R}$. Therefore the orbit of $\mathbb{R}^*$ through $(x, b, a) \in LS^1 \times \mathbb{R}$ is

$$[(x, b, a)] = \{(x, bg, g^{-1}a) | g \in \mathbb{R}^*\}$$

which we can identify with $(x, a) \in T_xS^1$ because as g runs through all of $\mathbb{R}^*$, so bg runs through all bases for T_xS^1.

Example 3. The tangent bundle is a special case of the *tensor bundles* $T_h{}^kM$. For all $k, h = 0, 1, \ldots$ these again have $P = LM$, but F is a tensor product of k copies of R^n and h copies of its dual $\mathbb{R}^{n*}$. Here $G = Gl(n; \mathbb{R})$ acts independently on the factors of the tensor product, as for TM on $\mathbb{R}^n$ and via transposed inverses on $\mathbb{R}^{n*}$. We conventionally identify $T_0{}^0M$ with $M \times \mathbb{R}$, $T_0{}^1M$ with TM, and $T_1{}^0M$ with TM^*, the *cotangent bundle.* When necessary we shall denote the projection maps onto M by $\Pi_h{}^k$. Now we can formulate a *tensor field* of type $\binom{k}{h}$ as a (smooth) section of $\Pi_h{}^k$. That is, some $w: M \to T_h{}^kM$ such that $\Pi_h{}^k \circ w = I_M$. The set of such form a vector space $T_h{}^kM$ for each k, h (see 2.1). Such smooth sections always exist for $T_h{}^kM$, as witness the zero tensor field of any type. However, the frame bundle LM need not have any smooth section, as witness the case $M = S^2$ in consequence of the hairy ball theorem. In fact there does not even exist a *continuous* section of TS^2 that is never zero.

4.4. Let (P, G, M) be a principal fibre bundle. For all $u \in P$ the *vertical subspace* G_u of the tangent space T_uP is given by the kernel of $D\Pi_P$ as

$$G_u = \{X \in T_uP | D\Pi_P(X) = 0 \in T_{\Pi_P(u)}M\}$$

Now we see that the map in 4.2(i) actually induces an *isomorphism*

$$\mathfrak{g} \to G_u: \gamma \mapsto \gamma_u{}^*$$

so $\dim G_u = \dim G$.

Example. For $P = LS^1$, $G = \mathbb{R}^*$ we have for all $u = (x, b) \in LS^1$

$$\begin{aligned}\mathbb{R}^*_{(x,b)} &= \{(x, b, p, q) \in T_{(x,b)}LS^1 | (x, p) = 0 \in T_xS^1\} \\ &= \{(x, b, 0, q) \in T_{(x,b)}LS^1 | q \in \mathbb{R}\} \cong \mathbb{R}\end{aligned}$$

4.5. *The canonical one-form* of a frame bundle LM is the map

$$\Theta: TLM \to \mathbb{R}^n: (u, X) \mapsto \Pi_u \circ D\Pi_L(u, X)$$

where for all $u = (x, (B_i)) \in LM$

$$\Pi_u: T_xM \to \mathbb{R}^n: a^iB_i \mapsto (a^i)$$

Since all vertical vectors lie in the kernel of $D\Pi_L$, $\Theta(G_u) = 0$ for all u.

Example. $\Theta: TLS^1 \to \mathbb{R}^1: (x, b, p, q) \mapsto p/b$. Or more generally, with matrix components,

$$\Theta: TLM \to \mathbb{R}^n: (x^i, b_j^i, X^i, B_j^i) \mapsto (b_j^i)^{-1}(X^i)$$

4.6. Universal covering manifold. Homotopic curves. Given a connected manifold M, there is a unique universal covering manifold $\tilde{M}$. [That is, a unique universal covering space (see 1.13) with manifold structure.]

It turns out that $(\tilde{M}, \pi_1(M), M)$ is a principal fibre bundle over M with structure group $\pi_1(M)$, the *first homotopy group* of M. ■

For the proof see Steenrod [60]), pp. 67–71, where the isomorphism classes implied by the uniqueness are also elaborated.

Example. Take $S^n = \{x \in \mathbb{R}^{n+1} | \ \|x\|_0 = 1\}$ for $n > 1$. Real projective n-space, $\mathbb{R}P^n$ is the quotient of S^n by the group

$$G = \{I_{S^n}, J\} \quad \text{where } J: x \mapsto -x$$

It turns out that $S^n = \tilde{\mathbb{R}P^n}$; the sphere is the universal covering manifold of projective space.

We do not wish to develop details of the homotopy group here (see Pontryagin [51], ch. 9). However, in 5.7 we do have occasion to use the idea of *homotopy* for closed curves. Essentially the curves c_1, c_2 are homotopic if one can be continuously deformed into the other. More precisely, two closed curves $c_1, c_2: [a, b] \to M$ with $c_1(a) = c_2(a) = x$ are *homotopic* if there exists a continuous map $f: [a, b] \times [0, 1] \to M$ such that

$$f(a, t) = f(b, t) = x, \qquad \text{for all } t \in [0, 1]$$

$$f(s, 0) = c_1(s), \qquad f(s, 1) = c_2(s), \qquad \text{for all } s \in [a, b]$$

A curve is *homotopic to zero* if it is homotopic to a constant curve.

5. CONNECTIONS

The earliest formulation of a connection on a manifold was by Weyl in synthesizing parallel transport of vectors along a curve. This was highly motivated by applications to space–times in relativity theory where it is essential to be able to compare vectors from different tangent spaces. Connections give rise to geodesics, covariant derivatives, and curvature; and in any particular case there is no escape from calculating their manifestation in coordinates, the Christoffel symbols. However, it suits our purpose to adopt a more global characterization to exploit the neat way that connections partition the geometry of principal fibre bundles generated by Lie groups. Expressions in local coordinates will come soon enough.

5.1. A *connection* ∇ in a principal fibre bundle (P, G, M) is an assignment of a subspace H_u of T_uP for all $u \in P$ such that

(i) $T_uP = H_u \oplus G_u$ smoothly on TP (see 4.4):
(ii) $DR_g(H_u) = H_{R_g(u)}$, for all $g \in G$ (see 5.6, Example 2).

Thus ∇ is a smooth *distribution* satisfying (i) and (ii). We call H_u the *horizontal subspace* of T_uP or at $u \in P$. It turns out that $D\Pi_P : TP \to TM$ induces an isomorphism $H_u \cong T_{\Pi_P(u)}M$ for all $u \in P$. It is common also to speak of ∇ as a *connection* on M, where the context determines which principal fibre bundle ∇ is in.

Example 1. Let $P = LS^1$; then a *constant* connection in LS^1 is given by any $\lambda \in \mathbb{R}$ if for all $u = (x, b) \in LS^1$ we put (see [17])

$$H_{(x,b)} = \{(x, b, p, -\lambda bp) | p \in \mathbb{R}\}$$

The vertical subspaces $\mathbb{R}^*_{(x),b)}$ were given in 4.4, so for any $(x, b, p, q) \in T_{(x,b)}LS^1$ we have the decomposition

$$(x, b, p, q) = (x, b, p, -\lambda bp) \oplus (x, b, 0, q + \lambda bp)$$

Evidently this is a smooth decomposition on TLS^1. Given any $g \in \mathbb{R}^*$, the right action of it on LS^1 is

$$R_g : LS^1 \to LS^1 : (x, b) \mapsto (x, bg)$$

so its differential is

$$DR_g : TLS^1 \to TLS^1 : (x, b, p, q) \mapsto (x, bg, p, qg)$$

Therefore, as required, we have

$$DR_g(H_{(x,b)}) = H_{(x,bg)}$$

Finally, the differential of the projection Π_L is

$$D\Pi_L : TLS^1 \to TS^1 : (x, b, p, q) \mapsto (x, p)$$

which yields the desired isomorphism

$$H_{(x,b)} \to T_xS^1 : (x, b, p, -\lambda bp) \mapsto (x, p)$$

Example 2. We observe that Example 1 is equally valid if S^1 is replaced throughout by $\mathbb{R}$.

Note also that in both examples, $H_{(x,b)}$ only *looks* horizontal (in the standard embedding in $\mathbb{R}^3$ for LS^1 or in $\mathbb{R}^2$ for $L\mathbb{R}$) if $\lambda = 0$. Finally, in the standard coordinate for either manifold, we have the solitary, constant, Christoffel symbol $\Gamma^1_{11} = \lambda$.

5.2. Let (P, G, M) be a principal fibre bundle with connection ∇. Every vector field $w: M \to TM$ has a unique *horizontal lift* $w^\dagger: P \to TP$ such that for all $u \in P$

$$D\Pi_P(w^\dagger(u)) = w \circ \Pi_P(u) \quad \text{and} \quad w^\dagger(u) \in H_u \;\blacksquare$$

Example. Take the case of a connection in a frame bundle LM. Then ∇ induces a derivation on vector fields on M. Local coordinates about $x \in M$ induce a basis $(\partial_i)_y$ for T_yM for all y in some neighborhood of x. Locally a vector field $v: M \to TM$ has an expansion $v = v^i\partial_i$. The derivation induced by ∇ is locally given by

$$\nabla_{w^j\partial_j}(v^i\partial_i) = (w^j\partial_j v^k + w^j v^i \Gamma^k{}_{ji})\partial_k$$

where

$$\Gamma^k{}_{ji}\partial_k = \nabla_{\partial_j}(\partial_i), \quad \text{for all } i, j$$

are fields having components the Christoffel symbols. Now a field $LM \to TLM: (x, b) \mapsto (x, b, X, B)$ turns out to be horizontal if and only if in local coordinates (see 5.6, Example 2)

$$B_j{}^i = -b_j{}^k X^l \Gamma^i{}_{kl}$$

This says precisely that each member $b_j{}^k\partial_k$ of the frame b at $x \in M$ satisfies the differential equation

$$(X(b_j{}^k) - B_j{}^k)\partial_k = \nabla_X(b_j{}^k\partial_k)$$

where of course the component X in (x, b, X, B) is interpreted as a tangent vector and hence a derivation at x.

We can write this symbolically in the form

$$B = X(b) - \nabla_X(b)$$

Hence given any vector field $w \in TM$ with

$$w: M \to TM: x \mapsto (x, X)$$

we obtain the horizontal lift $w^\dagger \in TLM$, with

$$w^\dagger: LM \to TLM: (x, b) \mapsto (x, b, X, X(b) - \nabla_X(b))$$

This is evidently unique, and since we have

$$D\Pi_L: TLM \to TM: (x, b, X, B) \mapsto (x, X)$$

the required projection property is achieved.

In particular for the constant connection λ on S^1, we find the horizontal lift

$$w^\uparrow : LS^1 \to TLS^1 : (x, b) \mapsto (x, b, X, -\lambda b X)$$

5.3. Horizontal lifts of vector fields have the following properties for all $v, w: M \to TM$:

(i) $(v + w)^\uparrow = v^\uparrow + w^\uparrow$;

(ii) For all smooth $f: M \to \mathbb{R}$ define $f^\uparrow = f \circ \Pi_P$; then $f^\uparrow \cdot v^\uparrow = (f \cdot v)^\uparrow$;

(iii) $[v^\uparrow, w^\uparrow]_H = [v, w]^\uparrow$ (H denotes horizontal component);

(iv) For all $g \in G$ and all $u \in P$

$$v^\uparrow(u) = v^\uparrow(R_g(u))$$

(v) Every horizontal vector field $\tilde{w}: P \to TP_H$ is the horizontal lift of some $w: M \to TM$. ■

We are particularly interested in the following consequence, on curves.

5.4. Every piecewise-C^1 curve $c: [0, 1) \to M$ has a unique horizontal lift $c^\uparrow: [0, 1) \to P$ such that the following is true:

$$c^\uparrow(0) = u_0 \in \Pi_P(c(0))$$

implies

$$\Pi_P \circ c^\uparrow = c$$

and the tangent vector field $\dot{c}^\uparrow$ is the horizontal lift of c. Then the map

$$\tau_t : \Pi_P^{\leftarrow}(c(0)) \to \Pi_P^{\leftarrow}(c(t)) : u_0 \mapsto c^\uparrow(t)$$

is a diffeomorphism for all t, called *parallel transport* along c. It commutes with the action of G. ■

Example. Again take the case of a connection ∇ in a frame bundle LM, developed in 5.2. We denote the restriction of ∇ to a curve by $\mathbb{V}$ (see [16]). Then the required curve $c^\uparrow$ has tangent vector field $\dot{c}^\uparrow$ satisfying

$$\mathbb{V}_{\dot{c}}(c^\uparrow) = 0$$

In local coordinates this becomes

$$\frac{d}{dt}(c^\uparrow)^i + (c^\uparrow)^j \dot{c}^k \Gamma^i{}_{jk} = 0$$

For a constant connection $\lambda = \Gamma^1{}_{11} \in \mathbb{R}$ on S^1, consider the (very!) typical curve with $\alpha \in \mathbb{R}$

$$c: [0, 1) \to S^1 : t \mapsto \alpha t, \quad \text{so } \dot{c}(t) = \alpha \ \forall t$$

We have

$$c^{\uparrow}\colon [0, 1) \to LS^1\colon t \mapsto (\alpha t, b(t))$$

with the *function* $b\colon [0, 1) \to \mathbb{R}^*$ satisfying

$$\frac{db}{dt} + b\lambda\alpha t = 0$$

Taking the initial condition $b(0) = b_0$, we find

$$b(t) = b_0 e^{-\lambda\alpha t}$$

It follows that the parallel transport map is

$$\tau_t\colon \Pi_L^{\leftarrow}(0) \to \Pi_L^{\leftarrow}(\alpha t)\colon (0, b_0) \mapsto (\alpha t, b_0 e^{-\lambda\alpha t})$$

Thus for any $\alpha \neq 0$, while c proceeds around the circle S^1, its horizontal lift through $(0, b_0)$ spirals up or down the cylinder LS^1. In fact the spiral remains in one component (see 4.1, Example 3) because the function b cannot take the value zero. Our argument is again equally valid if S^1 is replaced throughout by $\mathbb{R}$.

5.5. A connection ∇ in a frame bundle LM determines *geodesic curves* $c\colon [0, 1) \to M$ as the solutions of

$$\nabla_{\dot{c}}(\dot{c}) = 0$$

which in local coordinates becomes

$$\frac{d\dot{c}^i}{dt} + \dot{c}^j\dot{c}^k\Gamma^i_{jk} = 0$$

It turns out that c is a geodesic if and only if it is the projection of an integral curve of one of the *standard horizontal vector fields* determined for all $(X^i) \in \mathbb{R}^n$ by (see Example, 5.2)

$$LM \to TLM_H\colon (x, b) \mapsto (x, b, X, X(b) - \nabla_X(b))$$

where X has components (X^i). ■

5.6. The *connection form* of a connection ∇ in a principal fibre bundle (P, G, M) is the smooth map (see 4.2, 5.1)

$$\omega\colon TP \to \mathfrak{g}\colon X_H \oplus X_G \mapsto \gamma, \quad \text{with } \gamma_u{}^* = X_G(u)$$

It has the following properties:

(i) For all $\gamma \in \mathfrak{g}$, $\omega(\gamma_u{}^*) = \gamma$.
(ii) $\omega(X) = 0 \Leftrightarrow X = X_H \in TP$.
(iii) For all $g \in G$ and all vector fields $X\colon P \to TP$ (see 3.4)

$$\omega \circ DR_g(X) = \mathrm{Ad}\,(g^{-1})\omega(X).$$

(iv) Connections and connection forms determine one another uniquely. ■

Example 1. Take the case of the constant connection λ in LS^1. We know from 5.1, Example 1, that any $X = (x, b, p, q)$ has the unique decomposition

$$X = X_H \oplus X_G = (x, b, p, -\lambda bp) \oplus (x, b, 0, q + \lambda bp)$$

Moreover, from 3.7 and the examples in 4.2 we have

$$\gamma^*(x, b) = (x, b, 0, \gamma b), \quad \text{for all } \gamma \in \mathbb{R}$$

Hence we require

$$\omega(x, b, p, q) = \gamma = (q + \lambda bp)/b$$

which is well defined because $b \in \mathbb{R}^*$. The properties (*i*)–(*iv*) are easily seen to hold for γ. We elaborate just property (iii).

Let $g \in \mathbb{R}^*$ and $X: LS^1 \to TLS^1: (x, b) \mapsto (x, b, p, q)$. For all $g \in G$, Ad (g^{-1}) is the identity map on fields on $\mathbb{R}^*$

$$\text{Ad}\,(g^{-1})\omega(X) = D(L_{g^{-1}}R_g)\omega(X) = \omega(X)$$

Also

$$DR_g: TLS^1 \to TLS^1: (x, b, p, q) \mapsto (x, bg, p, qg)$$

and by 3.3

$$\omega(X): \mathbb{R}^* \to T\mathbb{R}^*: a \mapsto (a, a(q + \lambda p))$$

Therefore we find that the composite

$$\omega \circ DR_g(X): \mathbb{R}^* \to T\mathbb{R}^*$$

satisfies, as required,

$$\begin{aligned} \omega \circ DR_g(x, b, p, q) &= \omega(x, bg, p, qg) \\ &= (qg + \lambda bgp)/bg \\ &= (q + bp)/b = \omega(x, b, p, q) \end{aligned}$$

Example 2. We further develop Example 1 in 4.2. From 5.2 we find expressions in local coordinates as follows. Let $u = (x^i, b_j^i) \in LM$. Then a typical tangent vector at u appears as $Y = (x^i, b_j^i, X^i, B_j^i)$ and is partitioned by a connection with components Γ^i_{jk} into the direct sum

$$Y = Y_H \oplus Y_G = (x^i, b_j^i, X^i, -b_j^k\Gamma^i_{kl}X^l) \oplus (x^i, b_j^i, 0, B_j^i + b_j^k\Gamma^i_{kl}X^l)$$

Now, in components, $\omega(Y) \in \mathfrak{gl}(n; \mathbb{R})$ is that matrix $(\gamma_j^i) = \gamma$ such that $(\gamma_j^i)_u{}^* = Y_G$. From Example 1, 4.2, we find that this requires

$$(\gamma_m{}^i b_j{}^m) = (B_j{}^i + b_j{}^k \Gamma^i_{kl} X^l)$$

Or in matrix form

$$\gamma = (B + b\Gamma X)b^{-1}$$

Next we observe that for all $(g_j{}^i) = g \in Gl(n; \mathbb{R})$

$$\begin{aligned} DR_g(Y) &= (x^i, g_m{}^i b_j{}^m, X^i, -g_m{}^k b_j{}^m \Gamma^i_{kl} X^l) \\ &\oplus (x^i, g_m{}^i b_j{}^m, 0, g_m{}^i B_j{}^m + g_m{}^k b_j{}^m \Gamma^i_{kl} X^l) \end{aligned}$$

Therefore $\omega(DR_g(Y))$ is that $(\eta_j{}^i) = \eta$:

$$(\eta_k{}^i g_m{}^k b_j{}^m) = (g_m{}^i B_j{}^m + g_m{}^k b_j{}^m \Gamma^i_{kl} X^l)$$

Or in matrix form

$$\begin{aligned} \eta &= (gB + gb\Gamma X)(gb)^{-1} \\ &= g(B + b\Gamma X)b^{-1}g^{-1} \end{aligned}$$

which by the linearity of the action of g gives, as required,

$$\eta = R_{g^{-1}} \circ L_g(\gamma) = \mathrm{Ad}\,(g^{-1})(\gamma)$$

Note that the expression for $DR_g(Y)$ demonstrates property 5.1(ii).

5.7. Let (P, G, M) be a principal fibre bundle with connection ∇. We use the term *curve* to mean a piecewise continuously differentiable one. The *loop space* at any $x \in M$ is denoted $C(x)$ and consists of all closed curves starting and ending at x. There is a natural product on such curves. For all $c \in C(x)$ we have by 5.3 the parallel transport isomorphism

$$\tau_c \colon \Pi_P{}^{\leftarrow}(x) \to \Pi_P{}^{\leftarrow}(x)$$

and it commutes with the action of G on the fibre. The set of all such $\{\tau_c | c \in C(x)\}$, forms a group $\Phi(x)$, the *holonomy group* of ∇ with reference point x. We can realize $\Phi(x)$ as a subgroup $\Phi(u)$ of G for any $u \in P$, as

$$\Phi(u) = \{g \in G | R_g(u) = \tau_c(u),\ \tau_c \in \Phi(x)\}$$

Equivalently, if $\sim$ is the equivalence relation "can be joined by a horizontal curve" on P,

$$\Phi(u) = \{g \in G | u \sim R_g(u)\}$$

Then for all $u \in P$ it follows that

(i) $\forall g \in G,\ \Phi(R_g(u)) = \mathrm{ad}\,(g^{-1})\Phi(u)$
(ii) $u \sim v \Rightarrow \Phi(u) = \Phi(v)$
(iii) $\Phi(u)$ is a Lie group with identity component $\Phi^0(u)$ where $\Phi^0(u) = \{g \in \Phi(u) | R_g(u) = \tau_c(u);\ c$ is homotopic to zero in $C(x)\}$ is called the *restricted holonomy group* of ∇ with reference point u. ■

Example 1. Consider $L\mathbb{R}$ with constant connection λ. From 5.4 we observe that for all $x \in \mathbb{R}$, if c is a closed curve in $C(x)$, then τ_c is the identity on $\Pi_L{}^{\leftarrow}(x)$. Now $\mathbb{R}^*$ acts freely on the frame bundle $L\mathbb{R}$ and so for all $(x, b) \in L\mathbb{R}$

$$\Phi(x, b) = \{g \in \mathbb{R}^* | R_g(u) = u\} = \{1\}$$

We know that horizontal curves in $L\mathbb{R}$ are of the form

$$c^{\uparrow}: t \mapsto (x + \alpha t, be^{-\lambda \alpha t})$$

Hence $(x, b) \sim (x', b')$ if and only if for some $\alpha t = r \in \mathbb{R}$

$$x' - x = r \quad \text{and} \quad b'/b = e^{-\lambda r}$$

Example 2. Here we see a departure of the geometry of LS^1 from that of $L\mathbb{R}$, in the form of a nontrivial holonomy group. Again use the constant connection λ. We take S^1 to be $\mathbb{R}$ modulo the integers, $\mathbb{Z}$. We still have trivial members of $C(x)$, for any $x \in S^1$, but now we also have those curves which may make one or more circuits of S^1, such as

$$c_k: [0, k] \to S^1: t \mapsto (x + t) \ (\text{mod } 1), \qquad k \in \mathbb{Z}$$

In fact the essential members of $\Phi(x)$ are in the set $\{\tau_{c_k} | k \in \mathbb{Z}\}$, and we know that these elements commute with all R_g because

$$\tau_{c_k}: \Pi_L{}^{\leftarrow}(x) \to \Pi_L{}^{\leftarrow}(x): (x, b) \mapsto (x, be^{-\lambda k})$$

Accordingly the holonomy group with reference point (x, b) is

$$\Phi(x, b) = \{e^{-\lambda k} \in \mathbb{R}^* | k \in \mathbb{Z}\}$$

The relation $\sim$ has the same form as in Example 1, but on LS^1 we have $(x' - x) \ (\text{mod } 1) = 0$ whenever $(x' - x) \in \mathbb{Z}$. Therefore in all fibres $\Pi_L{}^{\leftarrow}(x)$ we find

$$(x, b) \sim (x, be^{-\lambda k}), \qquad k \in \mathbb{Z}$$

We know from 3.4 that for all $g^{-1} \in \mathbb{R}^*$, $\text{ad}\,(g^{-1})$ is the identity map on $\mathbb{R}^*$, and so

$$\text{ad}\,(g^{-1})\Phi(x, b) = \Phi(x, b)$$

Also, as required,

$$\Phi(R_g(x, b)) = \Phi(x, bg) = \Phi(x, b)$$

The identity component is trivial

$$\Phi^0(x, b) = \{e^{-\lambda k} \in \mathbb{R}^* | k = 0\} = \{1\}$$

See the example in II.3.12, which has holonomy group $\mathbb{R}^*$.

5.8. Let (P, G, M) be a principal fibre bundle with connection ∇. For all $u \in P$ the *holonomy bundle* through u is the subbundle

$$P(u) = \{v \in P | u \sim v\}$$

with structure group $\Phi(u)$. Since $\sim$ is an equivalence relation, the holonomy bundles partition P into nonempty disjoint sets. Moreover, every $g \in G$ maps each horizontal curve into a horizontal curve; we have the isomorphisms

$$R_g: P(u) \to P(ug), \qquad \mathrm{ad}\,(g^{-1}): \Phi(u) \to \Phi(ug)$$

Example. For Example 2 in 5.7 we find

$$P(x, b) = \{(x + r, be^{-\lambda r}) | r \in \mathbb{R}\}$$

and its structure group is $\mathbb{Z}$.

5.9. The *Levi–Cività connection* of a Riemannian or pseudo-Riemannian manifold M with metric tensor field $\mathbf{g} \in T_2M$ is the unique connection in LM such that (i) parallel transport is always an isometry along curves in M, and (ii) for all vector fields $v, w: M \to TM$ (see 5.2)

$$\nabla_v w - \nabla_w v = [v, w] \quad \blacksquare$$

The first condition is referred to as *compatibility with* $\mathbf{g}$, the second is referred to as the *symmetry* or *torsion-free* property of the connection.

Example 1. The constant connection λ in $L\mathbb{R}$ is the Levi–Cività connection induced by the Riemannian metric tensor field given locally by $g_{11} = e^{2\lambda x}$ at $x \in \mathbb{R}$, that is,

$$\mathbf{g}_x: T_x\mathbb{R} \times T_x\mathbb{R} \to \mathbb{R}: ((x, y), (x, z)) \mapsto yze^{2\lambda x}$$

However, the constant connection λ in LS^1 does not arise as the Levi–Cività connection of any Riemannian metric on S^1.

Example 2. Consider the cylinder, useful in the sequel, given by

$$N = \{(\psi, \sigma \in \mathbb{R}^2 | \psi \in (0, 2\pi), \sigma \in [0, 2\pi)\} \simeq (0, 2\pi) \times S^1$$

where we have identified $\sigma = 0$ and $\sigma = 2\pi$. So S^1 appears as the real numbers modulo 2π. We can supply this with the pseudo-Riemannian metric tensor field $\mathbf{g}$ with coordinates in the above indicated chart given by the matrix (see [7] and [15])

$$(g_{ij}) = (1 - \cos\psi)^2 \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{at } (\psi, \sigma) \in N$$

From the well-known local coordinate form of this theorem, the components of the induced Levi–Cività connection are given by (see [16])

$$2g_{ml}\Gamma^m_{ij} = \partial_i g_{jl} - \partial_l g_{ij} + \partial_j g_{il}$$

And so we find at $(\psi, \sigma) \in N$

$$(\Gamma^1_{ij}) = \frac{\sin \psi}{1 - \cos \psi} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad (\Gamma^2_{ij}) = \frac{\sin \psi}{1 - \cos \psi} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

5.10. The *bundle of orthonormal frames* of a Riemannian or pseudo-Riemannian manifold $(M, \mathbf{g})$ is the subbundle OM of LM, consisting of orthonormal frames

$$OM = \{(x, (X_i)) \in LM \mid |\mathbf{g}_x(X_i, X_j)| = \delta_{ij}\}$$

Hence a connection in LM induces a connection in OM. ∎

Example. Consider the pseudo-Riemannian cylinder $(N, \mathbf{g})$ in the preceding example. The component O^+N of positively oriented orthonormal bases has the structure group mentioned in 3.1, Example 2. We readily observe that

$$\frac{1}{1 - \cos \psi} \begin{bmatrix} \partial_\psi \\ \partial_\sigma \end{bmatrix}$$

is an orthonormal basis for $T_{(\psi,\sigma)}N$. We can arrange to locate this at $\chi = 0$ (see 3.1); then

$$O^+N = \left\{ \left(\psi, \sigma, \begin{bmatrix} \cosh \chi & \sinh \chi \\ \sinh \chi & \cosh \chi \end{bmatrix} \frac{1}{1 - \cos \psi} \begin{bmatrix} \partial_\psi \\ \partial_\sigma \end{bmatrix} \right) \middle| \chi \in \mathbb{R} \right\}$$

Hence we can think of this bundle as

$$O^+N = \{(\psi, \sigma, \chi) \in (0, 2\pi) \times S^1 \times \mathbb{R}\} \simeq N \times \mathbb{R}$$

We see that $\chi \in \mathbb{R}$ determines at $(\psi, \sigma) \in N$ a basis $(\chi_1{}^j\partial_j, \chi_2{}^j\partial_j)$, which we have expressed with respect to the basis (∂_1, ∂_2) induced by the coordinates (ψ, σ), such that

$$\chi_1{}^1 = \chi_2{}^2 = \frac{\cosh \chi}{1 - \cos \psi}$$

$$\chi_2{}^1 = \chi_1{}^2 = \frac{\sinh \chi}{1 - \cos \psi} \qquad \text{(see [7])}$$

It follows from 5.4 that any curve $c\colon [0, 1] \to N$ has a unique horizontal lift to O^+N. This is given by

$$c^\dagger\colon [0, 1] \to O^+N\colon t \mapsto (c(t), \chi(t))$$

where the real *function* χ satisfies

$$\frac{d\chi}{dt} = -\dot{c}^2 \frac{\sin c^1}{1 - \cos c^1}$$

and this equation is implied by

$$\nabla_{\dot{c}}(\chi_i{}^j \partial_j) = 0, \qquad i = 1, 2$$

or equivalently

$$\frac{d}{dt}\chi_i{}^j = -\Gamma^j_{kl}\dot{c}^k \chi_i{}^l, \qquad i, j, k = 1, 2$$

where (the function) χ determines the (function) $\chi_i{}^j$ as before.

We find, for example, that

$$c: t \mapsto (\psi_0, t), \qquad \psi_0 = \text{constant}$$

has horizontal lift through $\chi(0) = \chi_0$ at $t = 0$ given by

$$c^{\uparrow}: t \mapsto (\psi_0, t, \chi_0 - t\alpha_0), \qquad \alpha_0 = \sin\psi_0/(1 - \cos\psi_0)$$

The corresponding parallel transport is therefore essentially

$$\tau_c: \Pi_0{}^{\leftarrow}(c(0)) \to \Pi_0{}^{\leftarrow}(c(t)): \chi_0 \mapsto \chi_0 - t\frac{\sin\psi_0}{1 - \cos\psi_0}$$

which is evidently an isometry for $\mathbf{g}$ by construction, and in consequence of the identity

$$\cosh^2\chi - \sinh^2\chi = 1$$

5.11. In a Riemannian or pseudo-Riemannian manifold $(M, \mathbf{g})$, for all $x \in M$ and all $v \in T_xM$ there is a unique geodesic curve c in M such that $c(0) = x$, $\dot{c}(0) = v$ (see 5.5).

The map $\exp_x$ is defined on the subset

$$E_x = \{v \in T_xM \mid \exists \text{ geodesic } c_v: [0, 1] \to M \text{ with } c_v(0) = x, \dot{c}_v(0) = v\}$$

by putting

$$\exp_x: E_x \to M: v \mapsto c_v(1)$$

It is always smooth on some neighborhood of $0 \in T_xM$. ∎

5.12. A connected Riemannian manifold $(M, \mathbf{g})$ is complete (see 2.4) if and only if it is *geodesically complete* with respect to the induced Levi–Cività connection. By geodesically complete we mean that every geodesic admits an extension to arbitrarily large parameter values. A connected complete Riemannian manifold has the following properties:

(i) For all $x \in M$, the map $\exp_x: T_xM \to M$ is surjective.
(ii) Any two points can be joined by a minimizing geodesic. ∎

5.13. A connected Riemannian manifold $(M, \mathbf{g})$ is complete if any of the following conditions hold:

(i) All geodesics starting from any particular point are complete.
(ii) M is compact.
(iii) The group of isometries is transitive on M (see 3.8).
(iv) The orbit of the group of isometries through any particular point in M contains an open set of M. [Then the open set coincides with M and we have (iii).] ■

5.14. The *curvature* of a connection ∇ in a frame bundle LM is the map on pairs of vector fields defined by

$$R\colon Y^1M \times Y^1M \to L(Y^1M; Y^1M)\colon (v, w) \mapsto R_{(v,w)}$$

$$R_{(v,w)}\colon Y^1M \to Y^1M\colon z \mapsto \nabla_v(\nabla_w z) - \nabla_w(\nabla_v z) - \nabla_{[v,w]}z$$

It follows that we can consider the curvature as a tensor field of type $\binom{1}{3}$, so $R \in Y_3{}^1M$. If ∇ is the Levi–Cività connection of some $\mathbf{g} \in Y_2M$, then we call R the *Riemann tensor*. The curvature has a variety of well-known geometrical attributes, and Riemann tensors in particular possess a range of symmetry properties. We shall recall these as necessary in the sequel (see [16] for a detailed account).

The *Ricci tensor* is the unique (up to sign) contraction of the Riemann tensor to a symmetric member of Y_2M. In the theory of relativity it is used to link the physical energy–momentum tensor to the geometry of space–time (see Part III). We fix the choice of signs by indicating local components:

Riemann components: $R^l_{ikj} = \partial_k\Gamma^l_{ij} - \partial_j\Gamma^l_{ik} + \Gamma^l_{mk}\Gamma^m_{ij} - \Gamma^l_{jm}\Gamma^m_{ik}$
Ricci components: $R_{ij} = R^k_{ikj}$
Scalar curvature: $g^{ij}R_{ij}$

Part II. Connection Geometry from Frame Bundles

We have seen how a connection in a frame bundle LM decomposes the tangent spaces into a direct sum. The horizontal component spans "directions in the manifold M" and the vertical component spans "directions in the fibre," that is, in $Gl(n; \mathbb{R})$. We recalled that a Riemannian or pseudo-Riemannian metric tensor field on M always determines the Levi–Cività connection in LM. In a space–time manifold M we have, implicitly determined from the disposition of matter, a Lorentz metric tensor field and hence a connection. Also, when it is convenient, we can work on a space–time with a subbundle of LM, namely the bundle OM of orthonormal frames with structure group the Lorentz group. Sometimes, to reduce the dimensions still further, we can deal with submanifolds and their bundles to gain useful information about the whole space–time.

In order not to confuse the geometry that arises from a connection with the geometry that requires a metric tensor field on the underlying manifold, we avoid specializing to space–times where appropriate. Thus we consider a manifold M with a connection ∇. This ∇ induces a Riemannian metric on the frame bundle LM in such a way as to make horizontal and vertical subspaces orthogonal. Here we are interested in the consequent geometry of LM and the Cauchy completion of its connected components. The precise definition of a space–time is given in Part III, where we consider the geometry induced by its Levi–Cività connection.

1. SCHMIDT'S BUNDLE METRIC

We suppose that M is a smooth n-manifold with a connection ∇ in LM. The ideas originate in Schmidt [55], though Hawking [34] has suggested an earlier origin in Ehresmann [22]. However, Schmidt was unaware of any influence of this work of Ehresmann on his construction.

1.1. Existence. We denote the standard inner product on $\mathbb{R}^m$ for any m by $\cdot$, and we recall the canonical one-form Θ from 4.5 of Part I and the connection form ω from 5.6 of Part I. Then the Riemannian metric defined by Schmidt on LM is denoted by $\langle\ ,\ \rangle$ where

$$\langle\ ,\ \rangle : TLM \times TLM \to \mathbb{R} : (X, Y) \mapsto \Theta(X)\cdot\Theta(Y) + \omega(X)\cdot\omega(Y)$$

Symmetry and bilinearity follow from that of $\cdot$ and the linearity of Θ and ω. Positive definiteness follows from that for $\cdot$ because for all $X \in T_uLM$

$$\|X\|^2 = \langle X, X\rangle = 0 \Rightarrow \Theta(x) = 0 \in \mathbb{R}^n \quad \text{and} \quad \omega(X) = 0 \in \mathbb{R}^{n^2}$$
$$\Rightarrow X \in G_u \quad \text{and} \quad X \in H_u \quad \text{(by I.4.4, 5.6)}$$
$$\Rightarrow X = 0 \ \blacksquare$$

Example. Take $M = S^1$ with the constant connection λ. From I.4.5, 5.6 for all $(x, b, p, q), (x, b, r, s) \in T_{(x,b)}LS^1$

$$\langle (x, b, p, q), (x, b, r, s)\rangle = (pr/b^2) + (q + \lambda bp)(s + \lambda br)/b^2$$
$$\|(x, b, p, q)\|^2 = (p/b)^2 + ((q + \lambda bp)/b)^2 \qquad \text{(see [17])}$$

We find the length with respect to this norm of two curves in L^+S^1.

Case 1. $c: t \mapsto (x_0, t)$, for fixed $x_0 \in S^1$ and $t \in [1, t_m]$. This is a *vertical* curve with tangent vector

$$\dot{c}(t) = (x_0, t, 0, 1); \qquad \|\dot{c}(t)\| = 1/t$$

Hence the length is (for $t_m \geqslant 1$)

$$\int_1^{t_m} \|\dot{c}(t)\|\, dt = \log t_m$$

Since this tends to infinity as $t_m \to \infty$, we see that all the fibres $\Pi_{L^+}^{\leftarrow}(x_0) \subseteq L^+S^1$ are infinite for $x_0 \in S^1$.

Case 2. $c: t \mapsto (-t \bmod 1, e^{\lambda t})$, for $t \in [0, t_m]$. Again this begins at basis 1, but by I.5.5 this curve is *horizontal*, and for $t_m \geqslant 1$ it necessarily meets all fibres. We find the tangent vector

$$\dot{c}(t) = (-t \bmod 1, e^{\lambda t}, -1, \lambda e^{\lambda t})$$

so

$$\|\dot{c}(t)\| = e^{-\lambda t}$$

Hence the length is (for $t_m \geqslant 0$)

$$\left|\int_0^{t_m} e^{-\lambda t}\, dt\right| = (1 - e^{-\lambda t_m})/|\lambda|$$

Plainly this tends to $1/|\lambda|$ as $t_m \to \infty$. So we have an example of a curve of finite length in L^+S^1 whose projection covers S^1 infinitely many times.

1.2. Uniqueness We prove the claim of Schmidt that if $\cdot$ is replaced by different inner products on $\mathbb{R}^n$, $\mathbb{R}^{n^2}$, then the ensuing metric structure for $L'M$ is uniformly equivalent to that given by $\langle\ ,\ \rangle$. (See I.2.4, 1.11.)

Proof. Consider the widest generalization of $\langle\ ,\ \rangle$. It will be of the form

$$\langle\!\langle\ ,\ \rangle\!\rangle : (X, Y) \mapsto \Theta(X) * \Theta(Y) + \omega(X) \circledast \omega(Y)$$

where $*$, $\circledast$ are inner products given by

$$*: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}: ((a^i), (b^i)) \mapsto g_{ij}a^i b^j$$

$$\circledast: \mathbb{R}^{n^2} \times \mathbb{R}^{n^2} \to \mathbb{R}: ((u^i), (v^i)) \mapsto G_{ij}u^i v^j$$

for some matrices of fixed numbers (g_{ij}), (G_{ij}).

Let d and d' be the topological metrics (see I.2.4) induced by $\langle\ ,\ \rangle$, $\langle\!\langle\ ,\ \rangle\!\rangle$ respectively on each connected component $L'M$. We must prove that for all $u, v \in L'M$

(i) $(\forall \epsilon_1 > 0)(\exists \delta_1 > 0): d(u, v) < \delta_1 \Rightarrow d'(u, v) < \epsilon_1$;

(ii) $(\forall \epsilon_2 > 0)(\exists \delta_2 > 0): d'(u, v) < \delta_2 \Rightarrow d(u, v) < \epsilon_2$.

From I.2.4 it is sufficient to consider the norms $\|\ \|$ and $\|\ \|_*$ induced by $\langle\ ,\ \rangle$ and $\langle\!\langle\ ,\ \rangle\!\rangle$, respectively, on an arbitrary tangent space $T_uL'M$. Suppose $X \in T_uL'M$ and $\Theta(X) = (h^i) \in \mathbb{R}^n$, $\omega(X) = (v^i) \in \mathbb{R}^{n^2}$. Then we have

$$\|X\|_*^2 = g_{ij}h^i h^j + G_{ij}v^i v^j$$

$$\|X\|^2 = \delta_{ij}h^i h^j + \delta_{ij}v^i v^j$$

We know that any two norms on a finite-dimensional vector space are uniformly equivalent. Hence there exist m_1, m_2, M_1, and M_2, all positive, such that for all $(h^i) \in \mathbb{R}^n$ and all $(v^i) \in \mathbb{R}^{n^2}$

$$m_1\delta_{ij}h^i h^j \leqslant g_{ij}h^i h^j \leqslant m_2\delta_{ij}h^i h^j$$

$$M_1\delta_{ij}v^i v^j \leqslant G_{ij}v^i v^j \leqslant M_2\delta_{ij}v^i v^j$$

Let $m_0 = \min\{m_1, M_1\}$, $M_0 = \max\{m_2, M_2\}$; then

$$m_0\|X\|^2 \leqslant \|X\|_*^2 \leqslant M_0\|X\|^2 \quad \blacksquare$$

1.3. Uniform Action of $Gl(n; \mathbb{R})$. For all $g \in G^+$, the identity component of $Gl(n; \mathbb{R})$, the right action R_g is uniformly continuous on the metric space $L'M$.

Proof. Previous results allow us to give a brief, complete proof that differs somewhat from that outlined by Schmidt [55]. First we observe that by I.2.4 the metric topology coincides with the manifold topology, so the proposition is well formed.

It is necessary to prove (see I.1.9) for all $g \in G^+$ and all $u, v \in L'M$ that

$$(\forall \epsilon > 0)(\exists \delta > 0): d(u, v) < \delta \Rightarrow d(R_g(u), R_g(v)) < \epsilon$$

From I.2.4 it is sufficient to show for all $Y \in T_uL'M$ and all $u \in L'M$ that

$$(\forall \epsilon > 0)(\exists \delta > 0): \|Y\| < \delta \Rightarrow \|DR_g(Y)\| < \epsilon$$

We follow example 2 in I.5.6 and write Y locally in matrix components as $Y = (x, b, X, B)$, so $DR_g(Y) = (x, gb, X, gB)$. From I.4.5 and I.5.6

$$\Theta(Y) = b^{-1}X, \qquad \Theta(DR_g(Y)) = (gb)^{-1}X = b^{-1}g^{-1}X$$
$$\omega(Y) = (B + b\Gamma X)b^{-1}$$
$$\omega(DR_g(Y)) = g(B + b\Gamma X)b^{-1}g^{-1}$$

This reduces the problem to the following geometrical result (see [16], pp. 132ff.).

Lemma. Suppose $f \in Gl(n; \mathbb{R})$. Then there exists $r > 0$ such that for all $X \in \mathbb{R}^n$

$$\|f(X)\|_0 \leqslant r\|X\|_0$$

Proof. Since f is linear, it is sufficient to work with unit vectors X. The unit ball is compact and f is continuous, so f is bounded as required. ■

Returning to our main proof, we find

$$\|Y\|^2 = \|b^{-1}X\|_0 + \|(B + b\Gamma X)b^{-1}\|_0^2$$
$$\|DR_g(Y)\|^2 = \|b^{-1}g^{-1}X\|_0 + \|g(B + b\Gamma X)b^{-1}g^{-1}\|_0^2$$

We apply the lemma twice, with $f = b^{-1}$ and $f = b^{-1}g^{-1}$, to obtain

$$\|b^{-1}g^{-1}X\|_0^2 \leqslant r\|b^{-1}X\|_0^2$$

Now we use the lemma with $f \in Gl(n^2; \mathbb{R})$ given by

$$f\colon \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}\colon W \mapsto gWg^{-1}$$

Hence we find

$$\|g(B + b\Gamma X)b^{-1}g^{-1}\|_0^2 \leqslant s\|(B + b\Gamma X)b^{-1}\|_0^2$$

We put these together with $m = \max\{r, s\}$ and deduce that

$$\|DR_g(Y)\|^2 \leqslant m\|Y\|^2$$

from which the uniform continuity of R_g follows. ■

Corollary: Uniform Extension to Completion. From I.1.10 every R_g has a uniformly continuous extension $\bar{R}_g$ to the Cauchy completion $(\bar{L}'M, \bar{d})$ in which the metric space $(L'M, d)$ is dense (see I.2.4).

The extension $\bar{d}$ is also well defined and unique by the uniqueness (see I.1.4) of limits in a metric space, which is of course always Hausdorff. For similar reasons $\bar{R}_g$ is uniquely determined because for all $u_0 \in \bar{L}'M$ there is a unique limit,

$$\lim_{u \to u_0,\, u \in L'M} R_g(u), \quad \text{in } \bar{L}'M$$

From I.1.10 we recall that R_g agrees with $\bar{R}_g$ on $L'M$. ■

Example. From the example in 1.1 we see that for any $g \in \mathbb{R}^+$

$$\|(x, b, p, q)\|^2 = (p/b^2) + (q + \lambda bp)^2/b^2$$

$$\|DR_g(x, b, p, q)\|^2 = \|(x, gb, p, gq)\|^2 = (p/b^2)/g^2 + (q + \lambda bp)^2/b^2$$

The geometry of $\bar{L}\mathbb{R}$ and $\bar{L}S^1$ with the metric tensor field $\langle\ ,\ \rangle$ induced by a constant connection $\lambda \in \mathbb{R}$ has been described by Dodson and Sulley [17]. It turns out that for $\lambda \neq 0$ the essential part of $L\mathbb{R}$ is uniformly equivalent to

$$\{(u, v) \in \mathbb{R}^2 \mid |v| > 1/|\lambda|\}$$

with the standard metric, under the map

$$\varphi: L\mathbb{R} \to \mathbb{R}^2: (x, b) \mapsto (\lambda x + \log|b|, (1/b + \operatorname{sgn}(b))/|\lambda|)$$

Horizontal curves appear in $\varphi(L\mathbb{R})$ as the lines $u =$ constant, and the fibre $\Pi_L{}^{\leftarrow}(x)$ is the curve given by

$$v = \pm(1 + e^{\lambda x - u})/|\lambda|$$

For any $g \in \mathbb{R}^*$ the right-action appears in $\varphi(L\mathbb{R})$ as

$$\varphi \circ R_g \circ \varphi^{-1}: (u, v) \mapsto (u + \log|g|, (v/g) + \operatorname{sgn}(v)(\operatorname{sgn}(g) - 1/g)/|\lambda|)$$

The length of the horizontal curve through $x = 0$, $b = b_0$ is $1/|\lambda b|$. Evidently, Cauchy sequences in $\varphi(L\mathbb{R})$ with limits on the lines $v = \pm 1/|\lambda|$ establish the latter as the boundaries of the two components of $L\mathbb{R}$. The extension of R_g to $\bar{L}\mathbb{R}$ is then given by the action on $(u, \pm 1/|\lambda|)$:

$$\varphi \circ \bar{R}_g \circ \varphi^{-1}: (u, \pm 1/|\lambda|) \mapsto (u + \log|g|, \pm \operatorname{sgn}(g)/|\lambda|)$$

We can use φ to define a uniform equivalence of the essential part of LS^1 with the cylinder obtained by identifying points (u, v) and $(u + \lambda k, v)$, $k \in \mathbb{Z}$. Here the boundaries of L^+S^1 and L^-S^1 appear as the circles $v = \pm 1/|\lambda|$. Plainly this action of R_g and its extension also applies to LS^1 by taking u modulo λ. For $\bar{L}^{\pm}\mathbb{R}$ and $\bar{L}^{\pm}S^1$ the boundary is itself an orbit of the identity component $\mathbb{R}^+$ of $\mathbb{R}^*$.

1.4. The *b*-boundary. Our preceding result and corollary show that the identity component G^+ of $Gl(n; \mathbb{R})$ acts on the right (see I.3.6) of each manifold $L'M$. Hence from I.3.9 the topological space $\bar{M} = \bar{L}'M/G^+$ is well defined, with

$$\Pi_{\bar{L}}: \bar{L}'M \to \bar{M}: u \mapsto \{\bar{R}_g(u) \mid g \in G^+\}$$

continuous, and coincident with $\Pi_{L'}$ on $L'M$. Thus we have $\Pi_{L'}(L'M) = M$ and we define the *b-boundary* of M, with the given connection, to be $\partial M = \overline{M}\backslash M$. From the previous results of this section ∂M may be nonempty; if so, then it is essentially unique. ■

We shall call $\overline{M}$ the *bundle-completion of M*, with given connection.

Example 1. For $\mathbb{R}$ with constant connection λ we have from Example 1.3

$$\bar{L}^+\mathbb{R} \simeq \{(u, v) \in \mathbb{R}^2 | v \geqslant 1/|\lambda|\}$$

The orbits of $g \in \mathbb{R}^+$ appear as the exponential curves with

$$v = (1 + e^{\lambda x - u})/|\lambda|$$

One curve corresponds to each fibre $\Pi_L^{\leftarrow}(x)$. For each $x \in \mathbb{R}$ such a curve approaches the boundary $v = 1/|\lambda|$ only as $u \to +\infty$. Also an open set consisting of such curves and containing the boundary $v = 1/|\lambda|$ must include all curves for x in an interval $(-\infty, a)$ for some $a \leqslant \mathbb{R}$. Hence $\bar{R} = \bar{L}^+\mathbb{R}/\mathbb{R}^+$ is homeomorphic to a half-closed interval.

Example 2. For S^1 with constant connection λ we find

$$\bar{L}^+S^1 \simeq \{(u, v) \in S^1 \times \mathbb{R} | v \geqslant 1/|\lambda|\}$$

The orbits of $g \in \mathbb{R}^+$ appear as exponential spirals on this half-closed cylinder, and again the b-boundary consists of just one point. However, unlike the case in Example 1, any neighborhood of $v = 1/|\lambda|$ in $\bar{L}^+S^1$ intersects every fibre. So the only neighborhood of the b-boundary point of S^1 is $\bar{S}^1$. Hence $\bar{S}^1$ is at most a T_0-space and therefore not Hausdorff (see I.1.1). We can see that, in u-v coordinates with the standard metric on the cylinder

$$n \mapsto (\lambda a, (e^{-\lambda n} + 1)/|\lambda|)$$

is a Cauchy sequence in the isomorph of L^+S^1. But it is less obviously Cauchy in the original x-b coordinates (see III.2.7)

$$n \mapsto (a, e^{\lambda n})$$

1.5. Conformal and Projective Boundaries. For Minkowski space–time Penrose [50] devised a conformal boundary and Eardley and Sachs [20] devised a projective boundary. Schmidt [57] showed that any conformal or projective structure on any manifold M defines a natural boundary, and in the case of Minkowski space it coincides with the earlier boundaries.

Definition 1. Two metric tensor fields $\mathbf{g}$, $\mathbf{g}'$ are *conformally related* if for some real function σ, $\mathbf{g}' = e^{2\sigma}\mathbf{g}$. This is an equivalence relation and the class $[\mathbf{g}]$ of a given $\mathbf{g}$ is called a *conformal structure*.

Definition 2. Two symmetric linear connections ∇, $\tilde{\nabla}$ are *projectively related* if every ∇-geodesic is a $\tilde{\nabla}$-geodesic. This is an equivalence relation, and the class $[\nabla]$ of a given ∇ is called a *projective structure.*

Proposition (Schmidt [57]). A conformal structure $[\mathbf{g}]$ on M determines a conformal boundary $\partial_c M$, and similarly a projective structure determines a projective boundary.

Outline of Construction. For all $\mathbf{a} \in [\mathbf{g}]$ let $\hat{O}$ be the structure group of the orthonormal bundle $OM_{\mathbf{a}}$. Then there is a subbundle of LM

$$[OM_{\mathbf{g}}] = \{(x, (X)_i) \in OM_{\mathbf{a}} | \mathbf{a} \in [\mathbf{g}]\}$$

with structure group $\hat{O} \times \mathbb{R}$ and projection $\Pi_{\mathbf{g}}: [OM_{\mathbf{g}}] \to M$.

Every $\mathbf{a} \in [\mathbf{g}]$ yields a Levi–Cività connection $\nabla_{\mathbf{a}}$ and a corresponding horizontal subspace. Thus at each point $u \in [OM_{\mathbf{g}}]$ the conformal structure $[\mathbf{g}]$ determines a family $[H_u]$ of horizontal subspaces of $T_u[OM_{\mathbf{g}}]$. The members of this family are related by the differentials of the conformal factors, via their connections. It turns out that the corresponding n-tuplets of standard horizontal vector fields (see I.5.5) are mapped among themselves, essentially by the addition group $\mathbb{R}^n$.

From these observations Schmidt finds a subset P of the frame bundle $L[OM_{\mathbf{g}}]$, with structure group $\mathbb{R}^n$ and projection $\Pi_P: P \to [OM_{\mathbf{g}}]$. He is able to show that P is a principal fibre bundle (see I.4.1) over M with projection $\Pi_{\mathbf{g}} \circ \Pi_P$ and structure group K obtained from derivatives of elements of the conformal group $\hat{O} \times \mathbb{R}^+$.

Now every $\mathbf{a} \in [\mathbf{g}]$ determines via $\nabla_{\mathbf{a}}$ a curvature tensor $R_{\mathbf{a}}$ which uniquely decomposes into a sum

$$R_{\mathbf{a}} = C_{\mathbf{a}} + S_{\mathbf{a}}$$

where $C_{\mathbf{a}}$ is the conformal tensor and $S_{\mathbf{a}}$ depends only on the Ricci tensor, with $C_{\mathbf{a}} = C_{\mathbf{b}}$ for all $\mathbf{b} \in [\mathbf{g}]$. Moreover we can always find a real function σ such that for given $x \in M$, $\mathbf{a} \in [\mathbf{g}]$

$$e^{2\sigma}\mathbf{a} = \mathbf{b} \quad \text{in } [\mathbf{g}]$$

with $\sigma(x)$, $d\sigma(x)$, and $S_{\mathbf{b}}(x)$ all null. Now this $\mathbf{b}$ defines a unique connection and its associated n-tuplet of standard horizontal vector fields on LM. The map

$$D\Pi_P: TP \to T[OM_{\mathbf{g}}]$$

allows us to lift such fields to become vector fields on P.

Finally, the structure group K acts uniformly continuously on P with respect to the topological metric available from the parallelization (see I.2.7) by horizontal vector fields. Hence there is admitted a unique extension of the action of K to the Cauchy completion $\bar{P}$. We then define $M_c = \bar{P}/K$ and $\partial_c M = M_c \backslash M$.

The construction of the projective boundary follows similar lines because a projective relation between standard horizontal vector fields is of similar type to a conformal relation. ■

2. THE METRIC TOPOLOGY OF $\bar{L}'M$

As a metric space, $\bar{L}'M$ is necessarily normal (see I.1.7) and therefore Hausdorff. By construction $\bar{L}'M$ is connected, locally connected, and arcwise connected. It also satisfies the second axiom of countability: the metric topology on $\bar{L}'M$ has a countable base. Here we shall collect some further consequences of our choice of metric. We shall make occasional use here and subsequently of the survey article on fibre bundles by Eells [21]. At the same time, two standard texts are invaluable reference works: Pontryagin [51] and Steenrod [60].

2.1. G^+ **acts transitively on fibres.**

Proof. First, G^+ acts freely on $L'M$ by construction as a principal fibre bundle and transitively on $L'M$ because $L'M$ consists of *ordered* bases (see I.3.8, 4.1). Now suppose that $\bar{x} \in \partial M$ so $\Pi_{\bar{L}'}^{\leftarrow}(\bar{x}) \subseteq \bar{L}'M \backslash L'M$. Hence we may suppose that there exists some $u_0 \in \Pi_{\bar{L}'}^{\leftarrow}(\bar{x})$. Then from 1.4

$$\Pi_{\bar{L}'}^{\leftarrow}(\bar{x}) = \{\bar{R}_g(u_0) | g \in G^+\}$$

If $u, v \in \Pi_{\bar{L}'}^{\leftarrow}(\bar{x})$ then we have for some $g, h \in G^+$

$$\bar{R}_g(u_0) = u, \qquad \bar{R}_h(u_0) = v$$

Therefore $u = \bar{R}_g \bar{R}_{h-1}(v)$. ■

2.2. $\Pi_{\bar{L}'}$ **is an open map.**

Proof (see Eells [21]). Suppose U is open in $\bar{L}'M$; we show that $\Pi_{\bar{L}'}(U)$ is open in $\bar{M}$. Any $g \in G^+$ is a homeomorphism, so $U = \bar{R}_{g-1}\bar{R}_g(U)$ is open and hence $\bar{R}_g(U)$ is open.

From 1.4, $\Pi_{\bar{L}'}(U)$ is open if and only if $\Pi_{\bar{L}'}^{\leftarrow}(\Pi_{\bar{L}'}(U))$ is open in $\bar{L}'M$. We show that

$$\Pi_{\bar{L}'}^{\leftarrow}(\Pi_{\bar{L}'}(U)) = \bigcup \{\bar{R}_g(U) | g \in G^+\}$$

This is a union of open sets and therefore open.

Suppose $a \in \bar{R}_g(U)$; then $a = \bar{R}_g(b)$ for some $b \in U$. Hence

$$\Pi_{\bar{L}'}(a) = \Pi_{\bar{L}'}(b) \qquad \text{so} \qquad a \in \Pi_{\bar{L}'}^{\leftarrow}(\Pi_{\bar{L}}(U))$$

Conversely, suppose $a \in \Pi_{\bar{L}'}^{\leftarrow}(\Pi_{L'}(U))$. Then $\Pi_{L'}(a) = \Pi_{L'}(b)$ for some $b \in U$. But from 2.1, G^+ acts transitively on fibres so $\bar{R}_g(b) = a$ for some $g \in G^+$. Hence $a \in \bar{R}_g(U)$. ■

2.3. Completeness of Fibres with the Induced Metric. Suppose that (v_n) is a Cauchy sequence in a fibre over $x \in M$, in the induced metric, with limit $v \in \bar{L}'M$. Then $v \in \Pi_{\bar{L}'}^{\leftarrow}(x)$.

Proof. By construction $\bar{L}'M$ is complete, so $v \in \Pi_{\bar{L}'}^{\leftarrow}(y)$ for some $y \in \bar{M}$. Suppose $y \neq x$. Distinct fibres are disjoint, so for some $r > 0$, denoting the extension of the metric to $\bar{L}'M$ by $\bar{d}$, we see that

$$\bar{d}(\Pi_{\bar{L}'}^{\leftarrow}(x), v) = r$$

Then the open ball $S(v, r/2)$ contains v but does not meet $\Pi_{\bar{L}'}^{\leftarrow}(x)$. Hence (v_n) does not meet this ball. That contradicts its convergence to v, so $x = y$. ■

Corollary. By the transitivity of G^+ on fibres,

$$(\forall n \in \mathbb{N})(\exists g_n \in G^+): R_{g_n}(v_n) = v$$

Hence, by the continuity of each R_{g_n},

$$\lim g_n = e, \qquad \text{the identity in } G^+ \;■$$

A stronger aspect of the completeness of fibres over M is in the next result. Note that here we are measuring distances by taking infima over *curves in the fibre* for the Cauchy condition, in contrast to the situation in 3.5.

2.4. Fibres in $L'M$ are homogeneous spaces. Every fibre $\Pi_{L'}^{\leftarrow}(x)$ is a complete Riemannian submanifold.

Proof. The metric tensor field $\langle\ ,\ \rangle$ restricts to $\Pi_{L'}^{\leftarrow}(x)$, which is therefore a Riemannian submanifold of $L'M$. Let $u \in L'M$. Now if $Y \in T_u\Pi_{L'}^{\leftarrow}(x)$ then Y is vertical, and so by Definition 1.1 $\|Y\| = \|(\omega Y)\|_0$, in the notation of 1.2. It follows from the proof in 1.3 that the set of maps

$$\{DR_g | g \in G^+\}$$

gives rise to a group of isometries on $T_u\Pi_{L'}^{\leftarrow}(x)$. Since G^+ acts transitively on such fibres, by 2.1, the fibres are homogeneous. It is known that homogeneous Riemannian manifolds are complete; see, for example, Kobayashi and Nomizu [41], p. 176 (cf. I.5.13). ■

Example. Consider L^+S^1 with the constant connection λ. Then a typical tangent vector to the fibre over $(x, b) \in L^+S^1$ is $Y = (x, b, 0, q)$. From our previous results we have for any $g \in \mathbb{R}^+$, $DR_g(Y) = (x, bg, 0, qg)$ and so

$$\|Y\| = |q|/b, \qquad \|DR_g(Y)\| = |qg|/bg = \|Y\|$$

For any $x \in S^1$ the fibre $\Pi^{\leftarrow}_{L^+}(x)$ is complete, with infinite length, in the induced metric (see 1.1). However, $\Pi^{\leftarrow}_{L^+}(x)$ is incomplete and finite when considered with the full metric which (see 3.5) allows points to be linked by horizontal curves that may leave $\Pi^{\leftarrow}_{L^+}(x)$.

2.5. Completion of Bundles to Which the Connection is Reducible. Friedrich [25] showed that the b-boundary can be constructed on every closed subbundle of LM to which a Levi–Cività connection ∇ is reducible. Hence the construction respects product structures. Also connection-preserving mappings between space–times admit continuous extensions to their completions. Therefore the group of affine transformations (preserving connections) and the group of isometries of a space–time act as topological transformation groups on its completion.

Definition 1. Let H be a closed Lie subgroup of $Gl(n; \mathbb{R})$ with natural injection $j: H \to Gl(n; \mathbb{R})$. A principal fibre bundle HM over M with structure group H and projection Π_H is called an *H-structure* on M if there exists a bundle morphism

$$u: HM \to LM$$

such that $\Pi_H = \Pi_L \circ u$ and for all $h \in H$, $v \in HM$

$$u \circ R_h(v) = R_{j(h)} \circ u(v)$$

Such a map u is an imbedding and can be considered as a natural injection on the closed submanifold $HM \subseteq LM$.

$$\begin{array}{ccc} HM & \xrightarrow{R_h} & HM \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ LM & \xrightarrow[R_{j(h)}]{} & LM \end{array}$$

Definition 2. If under the conditions of Definition 1 there exist connections ∇ on LM and ∇' on HM such that their connection forms satisfy (see I.5.6)

$$\omega \circ Du = \omega'$$

then the connection ∇ is said to be *reducible* to HM. In this case we can of course use ω' to construct the Schmidt metric on HM and so obtain a Cauchy completion $\overline{H}M$ and *b–H-boundary* $\partial_H M$ by

$$\overline{M}_H = \overline{H}M/H = M \cup \partial_H M$$

Now we can give a precise statement of Friedrich's result.

> *Theorem.* Let H be a closed Lie subgroup of $Gl(4;\mathbb{R})$. If HM is an H-structure on M to which a given Levi–Cività connection is reducible, then the completion $\overline{M}_H = M \cup \partial_H M$ is homeomorphic to $\overline{M} = M \cup \partial M$.

Outline of Proof. (Full details are given in [25].) With the notation of Definition 2, u exists, is uniformly continuous, and so admits a uniformly continuous extension

$$\bar{u}\colon \overline{HM} \to \overline{LM}$$

The required homeomorphism is provided by

$$f\colon \overline{M}_H \to \overline{M}\colon x \mapsto \Pi_{\bar{L}} \circ \bar{u} \circ \Pi_{\bar{H}}(x) \;\blacksquare$$

We may expect the result to generalize to closed subgroups of $Gl(n;\mathbb{R})$. In the other direction Friedrich pointed out a special case for a space–time manifold (M, g), where g is a Lorentz type metric (see Part III).

> *Corollary 1.* If (M, g) is a simply connected space–time, then the completion $\overline{M}_H$ constructed on the holonomy bundle is homeomorphic to $\overline{M}$.

Proof. Since M is simply connected, the holonomy group H is connected. Schmidt [54] has shown (cf. [25]) that the connected Lie subgroups of the Lorentz group are closed in $Gl(4;\mathbb{R})$. $\blacksquare$

We offer another proof of this corollary in III.2.8.

> *Corollary 2.* If (M, g) is a space–time, then the completion $\overline{M}_0$ constructed on the orthonormal frame bundle is homeomorphic to $\overline{M}$. $\blacksquare$

Completion via the orthonormal frame bundle is so much more convenient for space–times that in Part III we shall abbreviate $\overline{M}_0$ to $\overline{M}$ and $\partial_0 M$ to ∂M. The same is of course true for Riemannian manifolds, and the important result for those is given in 3.7.

3. THE QUOTIENT TOPOLOGY OF $\overline{M}$

For each connected component $L'M$ and its Cauchy completion $\overline{L'M}$ we have induced essentially the same quotient topology for $\overline{M} = \overline{L'M}/G^+$. In fact $\overline{M}$ has a semimetric topology contained in the quotient topology. By construction $\overline{M}$ is connected, locally connected, and arcwise connected. It is second countable but need not be locally compact nor more than T_0. If M is a Riemannian manifold, then the bundle completion $\overline{M}$ via the Levi–Cività connection coincides with the direct Cauchy completion; this in particular persuades us of the mathematical significance of the b-boundary.

3.1. The Semimetric Structure ρ. There is a semimetric ρ for $\overline{M}$ with

$$\rho(x, y) = \inf\{\bar{d}(a, b) | a \in \Pi(x_{\bar{L}'}^{\leftarrow}), b \in \Pi_{\bar{L}'}^{\leftarrow}(y)\}$$

and the ρ-topology is contained in the quotient topology.

Proof. Plainly ρ is a semimetric. It determines a topology from the open balls of radius r, center $x \in \overline{M}$, given by

$$S(x, r) = \{y \in \overline{M} | \rho(x, y) < r\}$$

The quotient topology is the family of sets

$$\{U \subseteq \overline{M} | \Pi_{\bar{L}'}^{\leftarrow}(U) \text{ is open in } \bar{L}'M\}$$

We show that the quotient topology contains the ρ-topology by proving that $\Pi_{\bar{L}'}^{\leftarrow}$ is continuous in the ρ-topology. Without loss of generality, consider an arbitrary $x \in \overline{M}$ and an open set A_x of the form

$$\begin{aligned} A_x &= \{\Pi_{L'}(b) \in \overline{M} | \bar{d}(b, \Pi_{\bar{L}'}^{\leftarrow}(x)) < \epsilon\} \\ &= \{y \in \overline{M} | \rho(y, x) < \epsilon\}, \quad \text{for some } \epsilon > 0 \end{aligned}$$

We find an open ball $B(a, r)$ around any $a \in \Pi_{\bar{L}'}^{\leftarrow}(A_x)$ with $\bar{d}(a, \Pi_{\bar{L}'}^{\leftarrow}(x)) = s$.
Since $s < \epsilon$ we can find a suitable $r > 0$ with $r < \epsilon - s$. Therefore

$$a \in B(a, r) = \{c \in \bar{L}'M | \bar{d}(c, a) < r\} \subseteq \Pi_{\bar{L}'}^{\leftarrow}(A_x)$$

because

$$\bar{d}(c, \Pi_{\bar{L}'}^{\leftarrow}(x)) \leqslant \bar{d}(c, a) + \bar{d}(a, \Pi_{\bar{L}'}^{\leftarrow}(x)) \leqslant r + s < \epsilon \quad \blacksquare$$

3.2. If $L'M$ is complete and ρ is a metric, then M is complete.

Proof (see Eells [21], p. 63). Suppose that $L'M$ is complete, and let (x_n) be a Cauchy sequence in $M = L'M/G^+$. Using a subsequence if necessary, we suppose that

$$\rho(x_n, x_{n+1}) < (1/2)^n \qquad (*)$$

Choose any $b_1 \in \Pi_{L'}^{\leftarrow}(x_1)$ and $b_2 \in \Pi_{L'}^{\leftarrow}(x_2)$ such that $d(b_1, b_2) < 1/2$. This is possible because by (*) we know that

$$d(\Pi_{L'}^{\leftarrow}(x_1), \Pi_{L'}^{\leftarrow}(x_2)) < 1/2$$

Choose $b_3 \in \Pi_{L'}^{\leftarrow}(x_3)$ such that $d(b_2, b_3) < (1/2)^2$ and so on. The sequence (b_n) so formed is Cauchy in $L'M$ and so by hypothesis has a limit there, b_∞, which is unique.

Define $x_\infty = \Pi_{L'}(b_\infty) \in M$. Then we have

$$\rho(x_n, x_\infty) = \rho(\Pi_{L'}(b_n), \Pi_{L'}(b_\infty)) \leqslant d(b_n, b_\infty)$$

Hence (x_n) converges to x. $\blacksquare$

We know from 1.3 that G^+ is not an isometry group for $\bar{L}'M$, though it does yield isometries on the individual fibres by 2.4. An isometric action would have caused the semimetric topology to coincide with the quotient topology (see Eells [21], p. 62). However, the action of G^+ is uniform, and thus it may be possible to exploit the isometric action on vertically separated points, in studies of bunches of fibres over appropriately "small" sets in $\bar{M}$.

3.3. $\bar{M}$ is not T_1 if the orbits of G^+ are not closed in $\bar{L}'M$.

Proof. We prove the contrapositive form. Suppose $\bar{M}$ is T_1. For all $x \in \bar{M}$, $\{x\}$ is closed, so $\bar{M}\backslash\{x\}$ is open in the quotient topology. Thus $\Pi_{\bar{L}'}^{\leftarrow}(\bar{M}\backslash\{x\})$ is open in $\bar{L}'M$. Hence $\bar{L}'M\backslash\Pi(\bar{M}_{\bar{L}'}^{\leftarrow}\backslash\{x\}) = \Pi_{\bar{L}'}^{\leftarrow}\{x\}$ is closed. ∎

It is known (see I.1.3) that if $\bar{M}$ is T_1 in the semimetric ρ-topology then $\bar{M}$ is T_2 and ρ is a metric. However, our previous result does not preclude the existence of sets in $\bar{M}$ open in the quotient topology but not open in the ρ-topology.

From III.2.8 (for space–time M) we also obtain a homeomorph of $\bar{M}$ if we use the quotient by the holonomy group of the completed holonomy bundle. Hence the current and subsequent results concerning $\bar{M}$ have similar statements for that situation; see the example in 3.4.

3.4. $\bar{M}$ is $T_2 \Leftrightarrow$ graph G^+ is closed. $\bar{M}$ is Hausdorff if and only if graph G^+ is closed in $\bar{L}'M \times \bar{L}'M$.

Proof (see Kelley [39], p. 98; Eells [21], p. 61). The method is interesting. Note first that graph G^+ is the equivalence relation

$$\Delta = \{(u, \bar{R}_g(u)) | u \in \bar{L}'M, g \in G^+\}$$

and that $\bar{M} = \bar{L}'M/\Delta$. Also, the graph of $I_{\bar{M}}$

$$D = \{(x, x) | x \in \bar{M}\}$$

is an equivalence relation with $\bar{M}/D \simeq \bar{M}$ and

$$\Delta = (\Pi_{\bar{L}'} \times \Pi_{\bar{L}'})^{\leftarrow} D$$

(i) Suppose $\bar{M}$ is Hausdorff. If $(a, b) \notin \Delta$, then there exist disjoint neighborhoods U of $\Pi_{\bar{L}'}(a)$ and V of $\Pi_{\bar{L}'}(b)$ open in $\bar{M}$. (We use the quotient topology of course.) $\Pi_{\bar{L}'}^{\leftarrow}(U)$, $\Pi_{\bar{L}'}^{\leftarrow}(V)$ are therefore open in $\bar{L}'M$. Moreover, no point of one lies on a fibre that meets the other: they are not Δ-related. Hence $\Pi_{\bar{L}'}^{\leftarrow}(U) \times \Pi_{\bar{L}'}^{\leftarrow}(V)$ is an open neighborhood of (a, b), disjoint from Δ. So the complement of Δ is open and Δ is closed.

(ii) Suppose Δ is closed and $\Pi_{\bar{L}'}(a) \neq \Pi_{\bar{L}'}(b) \in \bar{M}$. Then $(a, b) \notin \Delta$. But Δ is closed so there are open neighborhoods U, V of a, b respectively with no point of U in a fibre meeting V: they are not Δ-related. Hence $\Pi_{\bar{L}'}(U)$

and $\Pi_{L'}(V)$ are disjoint and open neighborhoods of $\Pi_{L'}(a)$ and $\Pi_{L'}(b)$ because by 2.2 $\Pi_{L'}$ is an open map. ■

> *Corollary.* If $\overline{M}$ is Hausdorff, then D is closed in $\overline{M} \times \overline{M}$. Conversely, since the identity relation gives rise to an open projection $\overline{M} \to \overline{M}/D \simeq \overline{M}$, if D is closed then $\overline{M}$ is Hausdorff. ■

Example. The holonomy bundle through $(c_0, b_0) \in L^+\mathbb{R}$ with constant connection $\lambda \neq 0$ has a completion that by I.5.7 and II.2.5 is uniformly equivalent to (see Example III.2.8 for details)

$$\{(\lambda x_0 + \log b_0, v) | v \geqslant 1/|\lambda|\}$$

The structure group Φ is trivial, so we have $\overline{M} \simeq (-\infty, 1/|\lambda|]$. Furthermore, if A is closed, then $\Phi(A) = A$ is closed, so graph Φ is closed: we expect $\overline{M}$ to be Hausdorff. Clearly this is the case for $(-\infty, 1/|\lambda|]$.

By taking $(\lambda x_0 + \log b_0)$ mod λ, we find the completion of the holonomy bundle through $(x_0, b_0) \in L^+S^1$ with constant connection $\lambda \neq 0$. In this case, however, the holonomy group Φ is the integers (see I.5.7) and therefore nontrivial. The orbit of Φ is an exponentially distributed copy of the integers which is not closed and so by 3.2 $\bar{S}^1$ is not expected to be T_1. Consider the solitary b-boundary point Ω (see Example III.2.8). Let A be an open set in the quotient topology for $\bar{S}^1$, with $\Omega \in A$. We shall see that A must contain all of $\bar{S}^1$ because the fibres in $L^+S^1(x_0, b_0)$ over A must contain a family homeomorphic to

$$\{[((\lambda x_0 + \log b_0) \bmod \lambda, v)] | \alpha > v \geqslant 1/|\lambda|\}$$

for some real $\alpha > 1/|\lambda|$. But $\log(v|\lambda| - 1) \to -\infty$; so as we have observed before, all fibres over $\bar{S}^1$ will be met by the inverse image of A. (See 1.4 and [16].) We have seen that each holonomy bundle here appears as a vertical line $v \in [1/|\lambda|, \infty)$ on the cylinder $S^1 \times [1/|\lambda|, \infty)$ in the standard metric. A typical closed set in this line is therefore $[1/|\lambda|, \alpha]$ for some $\alpha > 1/|\lambda|$. Its image by the integer group Φ is

$$\begin{aligned}\Phi[1/|\lambda|, \alpha] &= \bigcup_{k \in Z} [1/|\lambda|, \alpha e^{\lambda k} + (1 - e^{\lambda k})/|\lambda|] \\ &= [1/|\lambda|, \infty)\end{aligned}$$

which is not closed.

3.5. Incomplete Fibres in the Full Metric. Suppose that (u_n) is a Cauchy sequence in $L'M$ without limit there and such that $\Pi_{L'}(u_n)$ is contained in a compact subset of M. Then we have the following:

(i) Some $x_0 \in M$: $\Pi_{L'}^{\leftarrow}(x_0)$ is incomplete in the full metric.
(ii) $\overline{M}$ is at most T_0.

Proof. (Schmidt [55] established this result; we amplify his proof.)

(i) By compactness in M there exists an infinite subsequence (u_n') with $\Pi_{L'}(u_n')$ convergent to some x in M. Now if (u_n') eventually lies in $\Pi_{L'}^{\leftarrow}(x)$, then by 2.3 it has a limit there if we measure distances along curves wholly in this fibre. Here, however, we use the full metric. Suppose that $\Pi_{L'}^{\leftarrow}(x)$ is complete in this metric. Then it is closed in $L'M$, and (u_n) has a limit in $\Pi_{L'}^{\leftarrow}(x)$ if it eventually lies in this fibre. By hypothesis this is not so, and hence we may suppose that for some $N \in \mathbb{N}$

$$d(u_n', \Pi_{L'}^{\leftarrow}(x)) > 0 \quad \text{for } n > N$$

We can find $v_n \in \Pi_{L'}^{\leftarrow}(x)$ such that for $n > N$

$$d(u_n', v_n) = d(u_n', \Pi_{L'}^{\leftarrow}(x))$$

Now, $\lim \Pi_{L'}(u_n') = x \Rightarrow d(u_n', v_n) \to 0$ as $n \to \infty$. Therefore, (u_n'), (v_n) are in the same equivalence class of the Cauchy completion; (v_n) is Cauchy because as $n, k \to \infty$

$$d(v_n, v_k) \leqslant d(v_n, u_n') + d(u_n', u_k') + d(u_k', v_k) \to 0$$

It follows that

$$\lim (v_n) = \lim (u_n') \in \bar{L}'M \backslash L'M$$

so $\Pi_{L'}^{\leftarrow}(x)$ is not complete because $\Pi_{L'}(\lim (u_n')) = x$.

(ii) From (i), $\Pi_{L'}^{\leftarrow}(x)$ contains a Cauchy sequence (v_n) without limit there. Suppose $\lim (v_n) = v \in \bar{L}'M \backslash L'M$. Then for all neighborhoods U of v we have

$$U \cap \Pi_{L'}^{\leftarrow}(x) \neq \varnothing$$

because v lies in the boundary of $\Pi_{L'}^{\leftarrow}(x)$. So if V is open in $\bar{M}$ with $\Pi_{L'}(v) \in V$, then also $x \in V$ so $\bar{M}$ is at most T_0. ■

Example. Consider L^+S^1 with constant connection $\lambda \neq 0$. Evidently S^1 is compact and contains the projection of the Cauchy sequence

$$(v_n)\colon \mathbb{N} \to L^+S^1 \colon n \mapsto ((x_0 - n) \bmod 1, e^{\lambda n})$$

That this is Cauchy is clear because it lies on the finite (horizontal) curve in Case 2, Example 1.1, and so

$$d(v_n, v_k) \leqslant \left| \int_n^k e^{-\lambda t}\, dt \right| = |e^{-\lambda k} - e^{-\lambda n}|/|\lambda|$$

We know that the essential part (see 1.3 and [17]) of L^+S^1 is uniformly equivalent to

$$W = \{(u, v) | u \in S^1, v > 1/|\lambda|\}$$

with the standard metric. There our Cauchy sequence appears as

$$n \mapsto (\lambda x_0 \bmod \lambda, (1 + e^{-\lambda n})/|\lambda|)$$

which has limit

$$(\lambda x_0 \bmod \lambda, 1/|\lambda|) \in \overline{W} \backslash W$$

Now as we see from its definition, (v_n) does lie wholly in the fibre $\Pi_{L'}^{\leftarrow}(x_0)$ for any $x_0 \in S^1$, so every fibre of L^+S^1 is incomplete in the full metric. Schmidt [55] gave an example of incomplete fibres arising from a two-dimensional Lorentz manifold.

3.6. ***M*** **is geodesically complete if** ***L'M*** **is complete.**

Proof (see Schmidt [55]). Suppose that $(L'M, \langle\ ,\ \rangle)$ is a complete Riemannian manifold. Then every Cauchy sequence is convergent in $L'M$, in particular so are those on horizontal curves (see III.2.7). In consequence any horizontal curve of finite length has endpoints in $L'M$.

From I.5.5 a curve in M is a geodesic if and only if it is the projection of an integral curve of one of the standard horizontal vector fields. By hypothesis these are complete, so geodesics in M can be extended to infinite parameter values. ■

The converse is false. Geroch [29] gave an example of a space–time manifold that was geodesically complete but contained an inextensible timelike curve of bounded acceleration.

3.7. If (M, g) **is a Riemannian manifold then its Cauchy completion is homeomorphic to its bundle completion with the Levi–Cività connection.**

Proof (see Schmidt [55]). We work with the bundle O^+M of orthonormal frames (cf. 2.5) and the distance function d_0 induced on it by inclusion in the Riemannian manifold $(L^+M, \langle\ ,\ \rangle)$. We also have a distance function d_g on M by I.2.4. It is sufficient to show that for all $x, y \in M$

$$d_0(\Pi_{O^+}^{\leftarrow}(x), \Pi_{O^+}^{\leftarrow}(y)) = d_g(x, y)$$

Suppose c is a curve in M with horizontal lift $c^\uparrow$ to O^+M. Then we have (see 1.2)

$$\|\dot{c}(t)\|_g = \|\dot{c}^\uparrow(t)\| = \|\Theta(\dot{c}^\uparrow(t))\|_0$$

because Θ effectively expresses $\dot{c}$ with respect to an orthonormal basis, and in such components $\|\ \|_g$ appears as the standard norm on $\mathbb{R}^n$. Thus $c^\uparrow$ has the same length as c. If c^* is a nonhorizontal curve in O^+M, that also projects onto c, then its length will be greater because the connection-form-term

$\|\omega(\dot{c}^*(t))\|_0$ is not then zero. The metrical equivalence of c and $c^\uparrow$ is independent of the choice of orthonormal frame through which the lift is made. The fibres are equidistant throughout their extent, and the result follows. ■

As pointed out by Schmidt, the coincidence of $\bar{M}$ with the Cauchy completion when it is available does encourage the view that it is the natural choice for general manifolds with connection. Certainly this seems to be true for pseudo-Riemannian cases, for it is a recent result of Stredder [59] that the metric tensor field determines the Levi–Cività connection merely by requiring that it constitute no additional structure, in the sense that it is natural with respect to restrictions. For the present we use 3.7 in establishing the following proposition and corollary.

3.8. $\bar{M}$ need not be locally compact.

Proof (Schmidt [55]). Let $\mathbb{R}^2$ have its standard metric structure, and define the subset

$$A = \{(x, \sin 1/x) | x \neq 0\} \cup \{(0, y) | \, |y| \leqslant 1\}$$

This A is closed, so $M = \mathbb{R}^2 \backslash A$ is a Riemannian manifold; we denote by M^- the connected component of $(0, -2) \in M$. By 3.7 we can effect the Cauchy completion and find the b-boundary

$$\partial M^- = \{(x, \sin 1/x) | x \neq 0\} \cup \{(0, -1)\}$$

Therefore $\bar{M}^- = M^- \cup \partial M^-$ is not locally compact because $(0, -1)$ has no compact neighborhood. ■

Corollary (Hawking and Ellis [35, p. 283]). Whereas the origin (0, 0) is in the topological boundary $\dot{M}^-$ of M^-, there is no curve in M^- with endpoint there and so the origin is not in the b-boundary (see I.1.12). ■

Consider the Riemannian submanifold

$$M = \mathbb{R}^2 \backslash \{(0, y) | \, |y| \leqslant 1\}$$

For each $(0, y)$ with $0 < y < 1$ there are two points in the b-boundary ∂M. Note that the manifold distance structure induced by the standard Riemannian metric on $\mathbb{R}^2$ differs from the usual topological metric, since it involves an infimum over curves in M. Hence the distance of $(1/n, 0)$ from $(-1/n, 0)$ tends to 2, for the minimizing geodesic must pass through $(0, 1)$ or $(0, -1)$. In contrast, the Euclidean distance between these points is $2/n$, which tends to zero.

3.9. b-incompleteness of curves in M. A curve c in M is said to have *finite bundle length* if it has a horizontal lift $c^\uparrow$ of finite length in $L'M$. This attribute is independent of the choice of point in $\Pi_L^\leftarrow(c)$ through which the lift is effected, for we have the property I.5.1(ii) of connections that assures us that the action of G^+ maps horizontal curves among themselves, and transitivity guarantees that the process is surjective; uniform continuity as displayed in 1.3 preserves the finiteness. (See [35], p. 259.)

A curve $c: [0, 1) \to M$ is called *b-incomplete* if it has finite bundle length and admits no continuous extension in M to domain [0, 1]. Evidently, the definition extends trivially to any (piecewise-C^1) reparameterization of the curve. From 1.4 it is clear that the b-boundary consists precisely of the endpoints in $\overline{M}$ of b-incomplete curves in M. We call M b-complete if $\partial M = \varnothing$, otherwise M is *b-incomplete.*

We have seen in 3.5 that a nonconvergent Cauchy sequence in $L'M$ whose projection is trapped in a compact set of M implies incomplete fibres and consequential loss of separation facilities in $\overline{M}$. Here we give a formulation in terms of b-incomplete curves, due to Hawking and Ellis.

3.10. Imprisoned b-incompleteness. A point $x \in M$ is not Hausdorff separated in $\overline{M}$ from a point $y \in \partial M$ if there is a b-incomplete curve c in M which has x as a limit point and y as an endpoint in $\overline{M}$.

Proof (Hawking and Ellis [35], p. 289). For the given curve c there is a horizontal lift $c^\dagger$ with an endpoint b which by hypothesis is such that

$$b \in \Pi_{\bar{L}'}^{\leftarrow}(y) \subseteq \bar{L}'M \backslash L'M$$

Let V be an open set, containing y, in $\overline{M}$. Then $\Pi_{\bar{L}'}^{\leftarrow}(V)$ is open in $\bar{L}'M$; since it contains b, it also contains $c^\dagger(t)$ for all t greater than some t_m. But then all points $c(t)$ for $t > t_m$ must lie in V. Hence V meets every neighborhood of x because x is a limit point of c. ■

We see that this type of boundary point generated by imprisoned b-incompleteness is qualitatively different from the situation of supplying points that previously were omitted when M was part of a larger (Hausdorff) manifold. Hawking and Ellis have pointed out that in Taub–NUT space–time (see [35], pp. 289, 170–178) there exist b-incomplete null geodesics totally imprisoned in compact sets.

3.11. A b-boundary contains the topological boundary. Let U be an open submanifold of M such that its closure U^c is compact in M. Then $\dot{U} \subseteq \partial U$.

Proof. Since U is a submanifold, $L'U$, $\bar{L}'U$, and ∂U are well defined. We may as well suppose $\dot{U} \neq \varnothing$. For all $x \in \dot{U} = U^c \backslash \text{int}(U)$, we have $x \notin U$ by the openness of U (see I.1.12). For the same reason we can find a curve $c: [0, 1) \to U$ with endpoint x.

Choose any $b_0 \in \Pi_{L'}^{\leftarrow}(c(0))$, and through it construct the unique horizontal lift $c^\dagger$. The latter curve possesses an endpoint

$$b_1 \in \Pi_{\bar{L}'}^{\leftarrow}(x) \subseteq \bar{L}'U$$

by compactness. Hence for all large enough $n \in \mathbb{N}$, $c^\uparrow$ eventually lies in the open ball

$$S(b_1, 1/n) \subseteq \bar{L}'U$$

Therefore, we can connect b_1 to some point on $c^\uparrow$ in $L'U$ by a minimizing geodesic of finite length. The projection of this geodesic is a b-incomplete curve in U, and since its endpoint is x we have $x \in \partial U$. Therefore $\dot{U} \subseteq \partial U$. ∎

Schmidt [56] has shown that every point in a space–time manifold (see Part III for definition) has an open neighborhood U such that $\dot{U} = \partial U$. This property, called *local b-completeness*, is a nontrivial consequence of the fact that every point in a space–time has a normal coordinate neighborhood in which no geodesic is imprisoned (see [41], p. 149; [35], p. 34).

3.12. ***M* is *b*-complete if *L′M* is complete.**

Proof. Suppose that $c\colon [0, 1) \to M$ is a b-incomplete curve. Let $c^\uparrow$ be its horizontal lift through some $b_0 \in \Pi_{L'}^{\leftarrow}(c(0))$. Now $c^\uparrow$ has finite length in $L'M$, but by continuity of $\Pi_{L'}$ it has no endpoint there. Hence it contains a nonconvergent Cauchy sequence. Thus if M is not b-complete, then $L'M$ is not complete. ∎

Example. Hawking and Ellis ([35], p. 278) proved that if M is a space–time, then a converse is also true (see III.1.1): a space–time M is b-complete if and only if O^+M is complete. This converse to 3.12 depends on the aforementioned local b-completeness that is enjoyed by space–times by virtue of their Lorentz metric. Schmidt [56] pointed out that on $\mathbb{R}^2$ there is a connection with respect to which no point is locally b-complete, namely, that ∇ which in the standard chart has components at $(x^1, x^2) \in \mathbb{R}^2$ given by

$$\Gamma^1_{11} = x^2, \qquad \Gamma^2_{22} = -x^1, \qquad \text{otherwise } \Gamma^i_{jk} = 0$$

The required property was established by the following geometrical argument, clearly of considerable power.

(i) $\mathbb{R}^2$ is simply connected, so by I.5.7 for all $u \in L^+\mathbb{R}^2$ we have $\Phi(u) = \Phi^0(u)$.

(ii) ∇ is analytic on the frame bundle and thus, by Kobayashi and Nomizu [41], p. 153, $\Phi^0(u)$ is determined by the successive covariant differentials of the curvature tensor (see I.5.14): R, ∇R, $\nabla^2 R, \ldots$, at $\Pi_{L'}(u)$.

(iii) At the origin in $\mathbb{R}^2$ we find $\nabla R = 0$, and there the only nonvanishing components of R are

$$R^1_{112} = R^2_{212} = 1$$

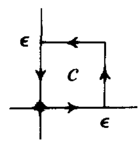
Hence the action of $\Phi(u)$ on tangent vectors at $\Pi_{L'}(u)$ is given by

$$\{R_\alpha\colon v \mapsto e^\alpha v \mid \alpha \in \mathbb{R}\}$$

(iv) We can choose a closed curve through the origin on which the parallel transport corresponds to R_α with $e^\alpha < 1$. Then for successive circuits of this curve the horizontal lift has monotonically decreasing tangent vector. Hence for infinitely many circuits the bundle length is finite and we have a b-incomplete curve. The process is applicable at all points. Moreover, arbitrarily small neighborhoods can contain b-incomplete curves, implying contributions to their b-boundary that have no counterpart in their topological boundary.

By way of illustration here, we shall calculate the holonomy group at the origin and display the b-incompleteness. Consider the following closed curve c, consisting of four parts c_1, c_2, c_3, and c_4, all with domain $[0, \epsilon]$ for some $\epsilon > 0$.

$$c: \begin{cases} c_1: t \mapsto (t, 0) & \dot{c}_1(t) = (1, 0) \\ c_2: t \mapsto (\epsilon, t); & \dot{c}_2(t) = (0, 1) \\ c_3: t \mapsto (\epsilon - t, \epsilon); & \dot{c}_3(t) = (-1, 0) \\ c_4: t \mapsto (0, \epsilon - t); & \dot{c}_4(t) = (0, -1) \end{cases}$$



Thus, the image of c is a square of side ϵ, with first corner $c_1(0) = (0, 0)$ and lying in the upper right-hand quadrant of $\mathbb{R}^2$. Evidently it is closed and homotopic to zero. Let $(A\partial_1 + B\partial_2, C\partial_1 + D\partial_2)$ be a basis for $T_{(0,0)}\mathbb{R}^2$; we find $c^\uparrow$, the horizontal lift of c through this point in $L^+\mathbb{R}^2$.

The required curve is given by (see I.5.4)

$$c^\uparrow: t \mapsto (c(t), \chi_i{}^j(t)\partial_j)$$

where the matrix of functions $\chi_i{}^j: [0, \epsilon] \to \mathbb{R}$ satisfies

$$\dot{\chi}_i{}^j = -\Gamma^j_{kl}\dot{c}^k\chi^l_i \quad \text{and} \quad \chi_i{}^j(0) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

We deduce the following solutions for the four parts.

$$c_1{}^\uparrow: t \mapsto \left(c_1(t), \begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$$

$$c_2{}^\uparrow: t \mapsto \left(c_2(t), \begin{bmatrix} A & Be^{\epsilon t} \\ C & De^{\epsilon t} \end{bmatrix}\right)$$

$$c_3{}^\uparrow: t \mapsto \left(c_3(t), \begin{bmatrix} Ae^{\epsilon t} & Be^{\epsilon^2} \\ Ce^{\epsilon t} & De^{\epsilon^2} \end{bmatrix}\right)$$

$$c_4{}^\uparrow: t \mapsto \left(c_4(t), e^{\epsilon^2}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$$

Hence the corresponding element of $\Phi \cong \mathbb{R}^+$ is e^{ϵ^2}, for each $\epsilon > 0$. (The inverse element corresponds to a curve with the same image as c but having a clockwise sense of direction around the square instead of counterclockwise.)

The bundle length of c is the length of $c^\dagger$, which is the sum of the lengths of the four parts (see 1.2)

$$\sum_{i=1}^{4} \int_0^{\epsilon} \|\dot{c}_i^\dagger(t)\| \, dt = \sum_{i=1}^{4} \int_0^{\epsilon} \|\Omega_k{}^j \dot{c}_i{}^k\|_0 \, dt$$

where $(\Omega_k{}^j) = (\chi_k{}^j)^{-1}$. Plainly this length is finite; we denote it by L_0.

Now consider the curve c_∞ which is the countably infinite composition of c with itself, corresponding to an unending series of circuits round the image of c. From the preceding comments and the definition of the bundle metric in 1.1, it is easy to see that the bundle lengths of successive circuits decrease with constant factor $e^{-\epsilon^2} = r$, say. It follows that the bundle length of c_∞ is

$$\sum_{k=0}^{\infty} L_0 r^k$$

which exists and is finite for $0 < r < 1$, by the property of geometric series. Thus c_∞ is a b-incomplete curve, in $\mathbb{R}^2$ with the given connection. Finally for any neighborhood of the origin in $\mathbb{R}^2$ we can choose $\epsilon > 0$ so that c lies in this neighborhood. Likewise for other points in $\mathbb{R}^2$. We note that, rather like the situation in Example 2, 1.4, the horizontal lift of c_∞ is an exponential spiral up the frame bundle, albeit on a square base here.

3.13. Projection of Finite Incomplete Curves. The projection of a finite incomplete curve in $L'M$ is a curve of finite bundle length.

Proof. We suppose that $L'M$ is incomplete and that there exists some (piecewise-C^1, as usual) curve

$$c^*: [0, 1) \to L'M$$

which admits no continuous extension to domain [0, 1] and which has finite length. We shall take the horizontal projection of c^* and show that it is the horizontal lift with finite length of the projection

$$\Pi_{L'} \circ c^* = c: [0, 1) \to M$$

The property I.5.1(i) of connections gives us a smooth decomposition of the tangent vector to the curve c^*, as

$$\dot{c}^*(t) = \dot{c}_H^*(t) \oplus \dot{c}_G^*(t)$$

Therefore we have a piecewise-continuous horizontal vector field $\dot{c}_H^*$ along c^*. By the fundamental theorem for ordinary differential equations the field

$\dot{c}_H^*$ has a unique piecewise-C^1 integral curve c_H with $\dot{c}_H = \dot{c}_H^*$ and $c_H(0) = c^*(0)$. By our construction it is clear that c_H has the same projection, c, as c^*.

$$\Pi_{L'} \circ c_H = \Pi_{L'} \circ c^* = c\colon [0, 1) \to M$$

Now every horizontal vector field on $L'M$ is the horizontal lift of some vector field on M [see I.5.3(v)]. Also, from the isomorphism property in I.5.1 we have

$$D\Pi_{L'}(\dot{c}^*(t)) = D\Pi_{L'}(\dot{c}_H^*(t)) = D\Pi_{L'}(\dot{c}_H(t)) = \dot{c}(t)$$

which by I.5.4 ensures that c_H is precisely $c^\dagger$, the unique horizontal lift of c through $c^*(0)$, perhaps with some constant sections included, where c^* was vertical.

The bundle length of c is (see 1.1, I.2.4)

$$\int_0^1 \|\dot{c}^\dagger(t)\| \, dt = \int_0^1 \|\dot{c}_H^*(t)\| \, dt$$

which cannot exceed the given finiteness of

$$\int_0^1 \|\dot{c}^*(t)\| \, dt$$

because

$$\begin{aligned}\|\dot{c}^*(t)\|^2 &= \|\Theta(\dot{c}^*(t))\|_0{}^2 + \|\omega(\dot{c}^*(t))\|_0{}^2 \\ &= \|\Theta(\dot{c}_H^*(t))\|_0{}^2 + \|\omega(\dot{c}_G^*(t))\|_0{}^2 \\ &= \|\dot{c}_H^*(t)\|^2 + \|\omega(\dot{c}_G^*(t))\|_0{}^2\end{aligned}$$

Thus c, the projection of the finite incomplete curve c^*, has finite bundle length. ■

To strengthen our result to a full converse of 3.12 we seem to need more structure than a mere connection. For example, Hawking and Ellis [35] have shown that in a space–time a projected curve of finite bundle length has no endpoint and is therefore b-incomplete. Their proof of the result corresponding to 3.13 differs from ours (see III.1.1). The sufficiency of a space–time in providing the extra structure required by Hawking and Ellis is implied by a lemma of Schmidt [56]. We can see what is needed by the following argument.

We try to contradict the supposition that c, in the proof of 3.13, has an endpoint x in M. Thus there is a continuous extension of c

$$\tilde{c}\colon [0, 1] \to M, \quad \text{with } \tilde{c}(1) = x$$

such that for all neighborhoods N of x there exists $t_N \in [0, 1)$ with

$$\tilde{c}(t) \in N \quad \text{for } t > t_N$$

We know that $\Pi_{L'}^{\leftarrow}(N)$ is open in $L'M$ and that it contains $c^{\dagger}(t)$ for $t > t_N$. If they exist in $L'M$, the limits

$$\lim_{t \to 1} c^{\dagger}(t) \quad \text{and} \quad \lim_{t \to 1} c^{*}(t)$$

must lie in $\Pi_{L'}^{\leftarrow}(x)$, which is contained in $\Pi_{L'}^{\leftarrow}(N)$, The Lorentz metric structure on a space–time allows us eventually to trap $c^{\dagger}$ and c^{*} in a compact subset of $\Pi_{L'}^{\leftarrow}(N)$ and so guarantee the existence of the limits to contradict the condition that c^{*} is without endpoint. This derives from Schmidt's proof [54] that the connected Lie subgroups of the Lorentz group are closed in $Gl(4; \mathbb{R})$. See Friedrich [25] for comments and applications to the case of simply connected manifolds for which the holonomy group is connected (cf. 2.5).

Part III. Geometrical Singularities in Space–Time

This section is designed to be complementary to chapter 8 in Hawking and Ellis [35], which thoroughly states the position in 1972 (see also Sachs and Wu [53]).

We suppose that *space–time* is a connected four-dimensional, Hausdorff oriented maximally extended smooth manifold M with a Lorentz metric tensor field **g**. It follows that $(M, \mathbf{g})$ is paracompact and hence regular and normal. The further physical requirements of local causality, local conservation, and Einstein's field equation (see 1.3) are developed in [35], chapter 3, and in [16], chapters 11 and 12 with more pictures and motivation. We further suppose that $(M, \mathbf{g})$ is time-orientable (see [35], p. 181; 2.1). There are then good reasons (see [35], p. 182) for believing that $(M, \mathbf{g})$ is also space orientable. In that case, simple connectedness is a sufficient condition for the existence of a global spinor field by Lee [46] and hence a parallelization (see I.2.7) by Geroch [27] (see 3.8, 4.3).

The geometry of space–time reflects physics by the curvature tensor R, so we certainly want it to be continuous (or perhaps just locally bounded, C^{0-}). Then Clarke [12] showed that every point has a neighborhood about which **g** can be expressed as C^2 (or C^{2-}) functions of some coordinates. For a discussion of the classification of curvature tensors via critical point theory and relations to the standard Petrov scheme, see Thorpe [61, 62].

The reasonable local properties that necessarily come with a Hausdorff topology and a Lorentz structure enabled Geroch [27] to establish paracompactness for space–times (see also [35]; 4.3). This deserves some comment, for by I.2.6 our space–time will therefore admit a Riemannian metric and so be metrizable (see [35], p. 38, for an explicit construction). However, there are many choices for this metric and only rarely might some be distinguished. Thus a Cauchy completion based on a metric induced by the paracompactness of the space–time is unlikely to be of physical significance in supplying substitutes for "singular points." On the other hand, Schmidt's metric derives from the connection whose curvature is controlled by the disposition of matter; so the physical relevance is assured, and it is reinforced by the essential uniqueness of the b-completion. Hence we see a strong case to proceed with the following definition.

By a *singularity* in M we mean a point in the b-boundary ∂M, induced by the Levi-Cività connection of **g** (see I.5.9, II.1.4). If M is bundle complete (see II.3.9), then it is geodesically complete (see I.5.13) but not conversely, by the example of Geroch [29]. The four pioneering singularity theorems of Penrose and Hawking give sufficient conditions for timelike or null geodesic incompleteness. Either would imply b-incompleteness and probably also points of unboundedly large curvature (see 1.3 for a statement of one of the theorems and 4.2 for Clarke's [10] result on the growth of curvature near a singularity.)

We know from II.1 that ∂M is calculated from $L'M$ by the factorization $\bar{L}'M/G^+$ where G^+ is the identity component of $Gl(4;\mathbb{R})$. However, for our space–time $(M, \mathbf{g})$ we can work with O^+M, the positively oriented component of the closed subbundle of orthonormal frames OM where (see II.2.5)

$$OM = \{(x, (X_i)) \in LM \,|\, \mathbf{g}_x(X_i, X_j) = g_{ij}\}$$

and (g_{ij}) is the diagonal matrix $(-1, 1, 1, 1)$ that fixes the signature of $\mathbf{g}$, in agreement with [35]. The structure group for OM is the *Lorentz group*, which we shall denote by $\hat{O}(1, 3)$ and which consists of all nonsingular 4×4 real matrices $(a_j^{\,i})$ such that (see Example 2, I.3.1)

$$a_i^{\,k} g_{kl} a_j^{\,l} = g_{ij}$$

This is a six-dimensional Lie subgroup of $Gl(4;\mathbb{R})$. We shall find it convenient to abbreviate $\hat{O}(1, 3)$ to $\hat{O}$ and denote its identity component by $\hat{O}^+$. Then the b-boundary is given, up to homeomorphism (see II.2.5, Corollary 2), by

$$\partial M = \bar{O}^+M/\hat{O}^+ \backslash M$$

This is well defined because we take the restriction to O^+M of Schmidt's metric $\langle\ ,\ \rangle$ for L^+M (see II.1). We shall denote the induced topological metric (see I.2.4) on O^+M and $\bar{O}^+M$ by d and $\bar{d}$, respectively. The canonical projections are denoted Π_{O^+} and $\Pi_{\bar{O}^+}$, respectively.

1. SUMMARY OF RESULTS IN HAWKING AND ELLIS [35]

The results on b-completeness available in 1972 were given by Hawking and Ellis [35], pp. 276–298, so we comment on them only briefly here. The most important is 1.1, which we mentioned in the examples of II.3.12. The others consider the character of singularities and analyze the situation of imprisoned b-incompleteness, which we met in II.3.10.

1.1. **$(M, \mathbf{g})$ is b-complete if and only if (O^+M, d) is complete.** A proof is given in [35], pp. 278–282. The sufficiency of the completeness of $(L'M, d)$ holds for any manifold with connection (see II.3.12). The necessity (for O^+M) depends on the existence of normal neighborhoods, of all points of $(M, \mathbf{g})$, wherein geodesics are not imprisoned (see II.3.11). The proof of this necessity is in two parts: Given a curve c^* in O^+M of finite length but without endpoint there, then $\Pi_{0^+} \circ c^* = c$, which is necessarily a curve in M, is such that

(i) c has finite bundle length: $\int \|\dot{c}\| < \infty$.
(ii) c has no endpoint in M.

In [35] the proof of (i) depends on the fact that any matrix $(a_j^i) \in \hat{O}(1, 3)$ can be decomposed as a product

$$(a_j^i) = \left[\begin{array}{c|c} A & \\ \hline & 1 \end{array}\right] \begin{bmatrix} \cosh\chi & 0 & 0 & \sinh\chi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\chi & 0 & 0 & \cosh\chi \end{bmatrix} \left[\begin{array}{c|c} B & \\ \hline & 1 \end{array}\right]$$

where A and B are 3×3 orthogonal matrices. This is also used in the proof of (ii), together with the compactness of the orthogonal group, following Schmidt [56]. We did of course prove a more general version of (i) in II.3.13, but there we wielded a fundamental theorem for the purpose.

1.2. The Character of the Singularities. The Penrose and Hawking singularity theorems show that a space–time (with physically reasonable properties) is geodesically incomplete if the metric g is of class C^2. However, b-incompleteness is well defined even if **g** is only of class C^{1-}. [By C^{r-} is meant that in local coordinates the rth partial derivatives of the components of g exist but satisfy only the Lipshitz condition, in the standard norm for coordinates.] Hawking and Ellis were able to show (see [35], pp. 284–289) that sufficient force persists in the theorems for a C^{2-} metric to preserve their results. See also Clarke [10] and 2.9.

Hence the singularities are likely to be points of unboundedly large curvature (unboundedly large tidal forces in physical terms) rather than mere discontinuities of curvature. Moreover, the evidence was fairly convincing that average values of curvature taken over any compact neighborhood of such a singular point would also be unbounded. Our firm belief in local causality and conservation leaves the implication that Einstein's field equation breaks down, together with the rest of normal physics, in the vicinity of these singularities.

Proposition (Hawking and Ellis [35], p. 290). Suppose that $x \in M$ is a limit point of a b-incomplete curve c (a case of imprisoned incompleteness; see II.3.10) and that at x (see 1.3)

$$R_{ij}K^iK^j \neq 0$$

for all nonspacelike vectors K. Then it follows that in some basis parallel propagated along c, some component of the curvature tensor becomes unbounded; so x is a p.p. curvature singularity (see [35], p. 260; 4.2). ■

This situation actually occurs in Taub–NUT space because it is devoid of matter. It led Hawking and Ellis to characterize such unrealistic behavior in the following way.

1.3. *b*-Boundedness of Space–Times. The idea here (see [35], p. 292) is to define a topology on a set of curves with a view to distinguishing those that display imprisoned b-incompleteness (see II.3.10). The method is to construct a covering space (see I.1.13 and I.4.6).

We define

$$B^+M = \{(c_v, v) | v \in L^+M;\ c_v \text{ is a } C^1 \text{ curve in } M, \text{ with no endpoint except } c_v(0) = \Pi_{L^+}(v)\}$$

and we suppose that each curve c_v is parameterized by bundle length with respect to the horizontal lift $c_v{}^{\uparrow}$ of c_v through v. For all open sets $U \subseteq M$ and $V \subseteq L^+M$ we define

$$U * V = \{(c_v, v) \in B^+M | c_v \text{ intersects } U \text{ and } v \in V\}$$

Such sets form (a subbasis for) our topology for B^+M.

We have a map analogous to that in I.5.11

$$\widetilde{\operatorname{Exp}}: B^+M \rightarrowtail M: (c_v, v) \mapsto c_v(1)$$

with domain contained in B^+M (see I.2). Plainly, if M is b-complete, $\widetilde{\operatorname{Exp}}$ will have domain B^+M; it is in any case continuous.

Hawking and Ellis defined M to be b-bounded if (see I.1.12)

$$W \text{ compact in } B^+M \Rightarrow (\widetilde{\operatorname{Exp}}\ W)^c \text{ compact in } M$$

This provides the desired separation of cases for the following reasons:

(i) If M is b-complete, then M is b-bounded, by the continuity of $\widetilde{\operatorname{Exp}}$ on the whole of B^+M.

(ii) Taub–NUT space–time (see [35], pp. 170–178, 289–292) is b-bounded but not b-complete.

Example. Support for our geometrical intuition can be found by contrasting the situations for $\mathbb{R}$ and S^1 with constant connection $\lambda \neq 0$. We know that both of these are b-incomplete; we show that $\mathbb{R}$ is not b-bounded but S^1 is b-bounded. (See II.1.4 for the b-boundaries.)

A family of elements in $B^+\mathbb{R}$ is generated by taking $m \in \mathbb{R}$ and (c_b, b) such that

$$(x, b) \in L^+\mathbb{R}, \qquad c_b: t \mapsto x + mt$$

Clearly this whole family is contained in a compact set $W \subset B^+\mathbb{R}$ induced by taking

$$x_1 \leqslant x \leqslant x_2, \qquad b_1 \leqslant b \leqslant b_2$$

But then $(\tilde{\mathrm{E}}\mathrm{xp}\ W)^c = \mathbb{R}$, which is not compact.

The corresponding situation for S^1 has a family

$$(c_b, b)\colon (x, b) \in L^+S^1, \qquad c_b\colon t \mapsto (x + mt) \bmod 1$$

This family is certainly typical; it covers S^1 with every member curve. But now the $\tilde{\mathrm{E}}\mathrm{xp}$ image of the compact set corresponding to W has compact closure.

We return to a space–time M to state the theorem of Hawking and Ellis [35], p. 292.

Theorem. A space–time M is *not* b-bounded if the following conditions hold:

(i) $R_{ij}K^iK^j \geqslant 0$, for every nonspacelike vector K.
(ii) There exists a compact spacelike three-surface S, without edge.
(iii) The unit normals to S are everywhere converging (or everywhere diverging) on S.
(iv) The energy–momentum tensor T is nonzero somewhere on S.
(v) $T_{ij}K^iK^j \geqslant 0$ and $T^{ij}K_i$ is zero or nonspacelike for every nonspacelike vector K, with $T_{ij}K^iK^j = 0$ only if $T^{ij}K_i = 0$.

Remark. Einstein's field equation is of course implicit in the physics of this result, linking the Ricci tensor to the energy–momentum tensor in coordinate form by (see I.5.14)

$$8\pi T_{ij} = R_{ij} - \tfrac{1}{2}Rg_{ij}, \qquad R \text{ is the scalar curvature}$$

However, the proof does not use this equation. It uses only the inequality (i), which amounts more or less to the requirement that gravity is attractive, a feature that any competing theory must share with general relativity (see also the proposition in 1.2).

The first three conditions are sufficient to establish that M is not timelike geodesically complete and hence not b-complete (see [35], p. 272). We note that local causality is not used in this theorem; violations of causality are insufficient to prevent the singularities.

Condition (iv) defeats the Taub–NUT space–time: it has $T = 0$ everywhere.

2. DEVELOPMENTS IN OTHER BUNDLES

We have defined the b-boundary via the frame bundle (see II.1.1), and we have seen how the metric structure for space–time $(M, \mathbf{g})$ yields the same b-boundary through the bundle of orthonormal frames. Now we follow

Sachs [52] in another homeomorphic formulation via a subbundle of the tangent bundle. There are two points to observe here. First, Sachs makes explicit use of the time-orientability that we have assumed for space–time. Second, in his bundle metric Sachs actually incorporates **g** itself; this is a stronger role than that of merely determining the Levi–Cività connection and may be more significant physically. Duncan and Shepley [19] have shown that the homeomorphism found by Sachs with the b-boundary can be simplified by modifying the classes of Cauchy sequences that determine it from the tangent bundle. The modification is in fact a hybrid scheme, using the Sachs metric to ascertain the Cauchy property and the Schmidt metric to compare limits, for sequences of unit timelike vectors. The main result of this section is again due to Schmidt: holonomy bundles generate the same b-boundary as frame bundles.

Hájiček and Schmidt [32] had previously shown how to extend the completion of the frame bundle to yield a completion of any associated bundle. Thus they obtain completions of all tensor bundles $T_h{}^kM$ and in particular of the tangent bundle. Moreover, they revealed the structure of the fibres of these bundles over boundary points in $\overline{M}$. In general these fibres do not have the vector space structure that is enjoyed by fibres over points of M. So though a limit of any tensor field can be found on ∂M, it will only be a multilinear function on nondegenerate fibres.

2.1. The Unit Future Subbundle $U^+M \subset TM$. The time-orientable property of $(M, \mathbf{g})$ provides a continuous partition of nonspacelike tangent vectors into two classes: *future directed* and *past directed.* (See Hawking and Ellis [35], ch. 6, devoted to causal structure for space–time.) The *unit future bundle* of $(M, \mathbf{g})$ is the subbundle

$$U^+M = \{(x, X) \in TM \mid X \text{ is unit timelike and future directed}\}$$

with projection $\Pi_U\colon U^+M \to M$, a surjection induced by the inclusion map $\iota\colon U^+M \hookrightarrow TM$.

Recall that YM is the space of all (TM-valued) smooth fields on M (see I.21; Example 3, I.4.3). We shall denote by YU^+M the space of all TM-valued smooth fields on U^+M. It is known (Bishop and Goldberg [5]) that YU^+M is spanned by the restrictions

$$\{w \circ \Pi_U \mid w \in YM\}$$

Use of this fact allows us to arrive at the same result as Sachs [52] but in a manner more consistent with our previous development.

The Levi–Cività connection ∇ induced by **g** gives a map (see I.5.2, 5.9) for all $x \in M$

$$\nabla\colon T_xM \times YM \to T_xM\colon (a, w) \to \nabla_a w$$

We can form a corresponding map

$$\nabla^*: T_X U^+M \times YU^+M \to T_xM: (A, w \circ \Pi_U) \to \nabla_{D\Pi_U A} w$$

for all $X \in U^+M$ with $\Pi_U(X) = x \in M$. In fact ∇^* is a *connection over* Π_U, in the terms of Bishop and Goldberg [5], p. 223. Thus $A \in T_X U^+M$ is

$$\text{vertical}^* \Leftrightarrow D\Pi_U A = 0$$

$$\text{horizontal}^* \Leftrightarrow \nabla_{D\Pi_U A}\iota = 0$$

where we have regarded the inclusion map $\iota: U^+M \hookrightarrow TM$ as an element of YU^+M.

Example. Consider the pseudo-Riemannian cylinder $(N, \mathbf{g})$ of Example 2 in I.5.9 (see Bosshard [7], Dodson [15]).

$$N = \{(\psi, \sigma) \in \mathbb{R}^2 | \psi \in (0, 2\pi), \sigma \in [0, 2\pi) \simeq S^1\}$$

$$\mathbf{g}: (g_{ij}) = (1 - \cos\psi)^2 \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{at } (\psi, \sigma) \in N$$

$$U^+N = \{(\psi, \sigma, r, s) \in TN | r = \sqrt[+]{s^2 + (1 - \cos\psi)^{-2}}, s \in \mathbb{R}\}$$

Here we have arbitrarily assigned "future directed" to "increasing ψ coordinate." We summarize the aforementioned maps, for $x = (\psi, \sigma)$ and $X = (\psi, \sigma, r, s) \in U^+N$. Let $A \in T_X U^+N$ be abbreviated to $(X, \mathbf{x}, \mathbf{a})$; so $\mathbf{x}$ is the component "along N" and $\mathbf{a}$ is the component "up TN," at $X \in U^+N$. We find the following:

$$D\Pi_U A = (x, \mathbf{x}) \in T_xN$$

$$\iota(X) = (x, (s^2 + (1 - \cos\psi)^{-2})^{1/2}, s) \in T_xN$$

$$\nabla_{D\Pi_U A}\iota = \nabla_{\mathbf{x}}\iota \in T_xN = x^k(\partial_k \iota^k + \Gamma^i_{jk}\iota^j)\partial_i$$

when $\mathbf{x} = x^k\partial_k$ and $\nabla_{\partial_j}\partial_k = \Gamma^i_{jk}\partial_i$ in a chart about x (see the example in I.5.2). Note that $D\Pi_U A$ need not be timelike, though by definition $\iota(X)$ must be. We shall see that $\nabla_{D\Pi_U A}\iota$ is always orthogonal to ι.

2.2. The Sachs metric for U^+M. There is a unique Riemannian metric $\hat{\mathbf{g}}$ on U^+M such that for all $A \in T_X U^+M$ and $\Pi_U(X) = x$, it agrees with the quadratic form given by

$$\hat{\mathbf{g}}_X(A, A) = \mathbf{g}_x(D\Pi_U A, D\Pi_U A) + 2(\mathbf{g}_x(D\Pi_U A, \iota(X)))^2 + \mathbf{g}_x(\nabla_{D\Pi_U A}\iota, \nabla_{D\Pi_U A}\iota)$$

Proof. We assume bilinearity; then the expansion of $\hat{\mathbf{g}}_X(A \pm B, A \pm B)$ yields $\hat{\mathbf{g}}_X(A, B)$ for all $A, B \in T_X U^+M$. Hence we have a bilinear form $\hat{\mathbf{g}}$, evidently smooth by the smoothness of the maps we are using, so $\hat{\mathbf{g}} \in Y_2 U^+M$. It remains to prove positive definiteness.

Since $\iota(X)$ is always unit timelike, we can choose an orthonormal frame (b_1, b_2, b_3, b_4) for T_xM with $\iota(X) = b_1$. Then for some real a^i we have $D\Pi_U A = a^i b_i$, and the first two terms in the expression for $\hat{\mathbf{g}}_X$ give

$$(-(a^1)^2 + (a^2)^2 + (a^3)^2 + (a^4)^2) + 2(-a^1)^2 \geqslant 0$$

with equality if and only if $a^i b_i = 0$. Hence the first two terms are positive definite, taken together.

Next we prove that $\nabla_{D\Pi_U A}\iota$ is a field always orthogonal to the field ι. But the Lorentz property of $\mathbf{g}$ guarantees positive definiteness on the orthogonal complement of a timelike vector field, so the desired result follows for $\hat{\mathbf{g}}$.

We use the compatibility of the Levi–Cività connection ∇ with $\mathbf{g}$ in the form [16]

$$\mathbf{x}(\mathbf{g}(\iota, \iota))_x = 2\mathbf{g}_x(\nabla_{\mathbf{x}}\iota, \iota)_x$$

In our case $\mathbf{x} = D\Pi_U A$, some derivation on smooth real functions about x (see I.2.1). However, the real function $\mathbf{g}(\iota, \iota)$ is constant because ι is a unit timelike field, so as required

$$\mathbf{g}_x(\nabla_{D\Pi_U A}\iota, \iota)_x = 0 \quad \blacksquare$$

The elements of this proof can be found in the article of Sachs [52], with some difficulty.

Example. We display this orthogonality for the pseudo-Riemannian cylinder $(N, \mathbf{g})$ of the previous example. The identity chart gives a basis $(\partial_1, \partial_2)_x$ for the tangent space T_xN at each $x = (\psi, \sigma) \in N$. We take

$$X = (x, (\sqrt{s^2 + (1 - \cos\psi)^{-2}})\, s) \in U^+N, \qquad D\Pi_U A = x^i\partial_i$$

and obtain the following expressions

$$\iota(X) = \sqrt{s^2 + (1 - \cos\psi)^{-2}}\,\partial_1 + s\partial_2$$

$$\nabla_{D\Pi_U A}\iota = \left(\frac{-x^1 \sin\psi(1 - \cos\psi)^{-3}}{\sqrt{s^2 + (1 - \cos\psi)^{-2}}} + \frac{\sin\psi}{(1 - \cos\psi)}(x^1\sqrt{s^2 + (1 - \cos\psi)^{-2}} + x^2 s)\right)\partial_1 + \left(\frac{\sin\psi}{(1 - \cos\psi)}(x^1 s + x^2\sqrt{s^2 + (1 - \cos\psi)^{-2}})\right)\partial_2$$

Application of $\mathbf{g}_x$ gives the required result.

2.3. The projection $\Pi_{OU}\colon O^+M \to U^+M$ is uniformly continuous. We suppose that (O^+M, d) is the metric space obtained via Schmidt's metric (see II.2.5 and the introduction to this part) and that (U^+M, d_Λ) is the metric space available (via I.2.4) from the Riemannian manifold $(U^+M, \hat{g})$ of Sachs. It is sufficient to establish uniform continuity (see I.1.9) if we show (see II.1.3) for all $b \in O^+M$ and all $Y \in T_bO^+M$ that

$$(\forall \epsilon > 0)(\exists \delta > 0)\colon \|Y\| < \delta \Rightarrow \|D\Pi_{OU} Y\|_\Lambda < \epsilon$$

Here $\| \ \|$ is the Schmidt norm and $\| \ \|_\Lambda$ is the Sachs norm induced by $\hat{g}$ on tangent vectors to U^+M.

Proof. The maps Π_{OU} and $D\Pi_{OU}$ are well defined and have the following local expressions using a chart about $x \in M$ giving rise to local basis fields (∂_i):

$$\Pi_{OU}\colon O^+M \to U^+M\colon (x, b_j{}^i\partial_i) \to (x, b_1{}^i\partial_i)$$
$$D\Pi_{OU}\colon TO^+M \to TU^+M\colon (x, b_j{}^i\partial_i, X^l, B_j{}^i) \to (x, b_1{}^i\partial_i, X^l, B_1{}^i)$$

Take the arbitrary $Y = (x, b_j{}^i, X^l, B_j{}^i) \in T_bO^+M$, the ∂_i in the second location can now be dropped. From Example 2, I.5.6, we know how Y splits into horizontal and vertical components, $Y = Y_H \oplus Y_G$. Then, as in II.3.13, we exploit the properties of Θ and ω to give

$$\|Y\|^2 = \|\Theta(Y_H)\|_0{}^2 + \|\omega(Y_G)\|_0{}^2$$

where (II.1.2) $\| \ \|_0$ is the standard norm on $\mathbb{R}^m$ for any m. It follows that

$$\|Y\|^2 = \|(b_j{}^i)^{-1}(X^l)\|_0{}^2 + \|(B_j{}^i + b_j{}^k\Gamma_{kl}^iX^l)(b_j{}^i)^{-1}\|_0{}^2$$
$$= \sum_{l=1}^{4} (\alpha^l)^2 + \sum_{i,j=1}^{4} (\beta_j^i)^2 \qquad (*)$$

where (α^l) and $(\beta_j{}^i)$ are the components of $X^l\partial_l$ and $B_j{}^i\partial_i$ with respect to the orthonormal frame $(b_j{}^i\partial_i)$.

We know, by Definition I.4.4, that $D\Pi_{OU}Y_G = 0$, so it is only necessary to calculate $\|D\Pi_{OU}Y_H\|_\Lambda$.

$$D\Pi_U \circ D\Pi_{OU}(Y_H) = (x, X^l)$$
$$\iota(b) = (x, b_1{}^i)$$
$$\nabla_{D\Pi_U \circ D\Pi_{OU}(Y_H)}\iota = (x, X^l\partial_l b_1{}^i - B_1{}^i)$$
$$\|D\Pi_{OU}Y\|_\Lambda{}^2 = g_{ij}X^iX^j + 2(g_{ij}X^ib_1{}^j)^2$$
$$+ g_{ij}(X^l\partial_l b_1{}^i - B_1{}^i, X^l\partial_l b_1{}^j - B_1{}^j)$$
$$= \sum_{l=1}^{4} (\alpha^l)^2 + \sum_{i=1}^{4} (\beta_1{}^i)^2 \qquad (**)$$

This is to be compared with (*). We see that

$$\|D\Pi_{OU}Y\|_\Lambda{}^2 < \|Y\|^2$$

and the result follows. ■

Corollary 1. We know that the metric space (O^+M, d) is dense in its Cauchy completion $(\bar{O}^+M, \bar{d})$ (see I.2.4). Also, the Cauchy completion $(\bar{U}^+M, \bar{d}_\Lambda)$ is well defined for the Riemannian manifold $(U^+M, \hat{g})$. Hence, by I.1.10 there exists a (unique by I.1.5) uniformly continuous extension of Π_{OU} given by

$$\Pi_{\bar{O}U}: \bar{O}^+M \to \bar{U}^+M, \qquad \Pi_{\bar{O}U}|_{O^+M} = \Pi_{OU} \quad \blacksquare$$

We have yet to prove that $\Pi_{\bar{O}U}$ is surjective.

Corollary 2 For all $u, v \in O^+M$

$$d_\Lambda(\Pi_{OU}(u), \Pi_{OU}(v)) \leqslant d(u, v) \quad \blacksquare$$

2.4. The extension $\Pi_{\bar{O}U}: \bar{O}^+M \to \bar{U}^+M$ is surjective.

Proof. Suppose not; then there exists a Cauchy sequence (y_n) in U^+M with limit y such that

$$y \in \bar{U}^+M \backslash \Pi_{\bar{O}U}(\bar{O}^+M)$$

Moreover, $\Pi_{\bar{O}U}$ is continuous by Corollary 1 in 2.3, so it preserves the limits of convergent sequences. Hence if (u_n) is a sequence in O^+M, with $\Pi_{OU}(u_n) = y_n$ for all n, then (u_n) is not Cauchy. For if it were, then it would converge in $\bar{O}^+M$ and its limit would necessarily project to y, contrary to hypothesis. This is the contradiction that we find: we lift (y_n) to a Cauchy sequence in O^+M.

The Cauchy property of (y_n) implies the existence of a family of curves $\{c_\nu: [0, 1) \to U^+M | \nu \in \mathbb{N}\}$ and a real sequence $(t_n) \subset [0, 1)$ convergent to 1 such that

(i) $(\forall \nu, n \in \mathbb{N})\ c_\nu(t_n) = y_n$
(ii) $(\forall \varepsilon > 0)(\exists N_\varepsilon \in \mathbb{N})$:

$$\lim_{\nu \to \infty} \left| \int_{t_n}^{t_k} \|\dot{c}_\nu(t)\|_\Lambda \, dt \right| = d_\Lambda(y_n, y_k) < \varepsilon, \qquad \text{for } n, k > N_\varepsilon$$

We seek a family of curves $\{c_\nu^*: [0, 1) \to O^+M | \nu \in \mathbb{N}\}$, such that $\Pi_{OU} \circ c_\nu^* = c_\nu$ and $D\Pi_{OU} \circ \dot{c}_\nu^* = \dot{c}_\nu$, for all $_\nu$. The forms of Π_{OU} and $D\Pi_{OU}$ were given in 2.3. So we readily find from

$$c_\nu(t) = (x(t), \hat{b}_1(t))$$
$$\dot{c}_\nu(t) = (x(t), \hat{b}_1(t), \alpha^i(t)\hat{b}_i(t), \beta_1{}^i(t)\hat{b}_i(t))$$

a suitable candidate c_ν^* with

$$\dot{c}_\nu^*(t) = (x(t), (\hat{b}_i)_t, \alpha^i(t)\hat{b}_i(t), (\beta_j{}^i\hat{b}_i)_t)$$

by choosing $\beta_j{}^i(t) = \beta_1{}^i(t)$ if $j = 1$ and $\beta_j{}^i(t) = 0$ if $j \neq 1$. With this choice we have for all $\nu \in \mathbb{N}$ and all $t \in [0, 1)$

$$\|\dot{c}_\nu^*(t)\|^2 = \|\dot{c}_\nu(t)\|_\Lambda{}^2$$

It follows that the curves c_ν^* and c_ν have the same length for all $\nu \in \mathbb{N}$.

In consequence, there exists the limit

$$\lim_{\nu\to\infty} \left| \int_{t_n}^{t_k} \|\dot{c}_\nu^*(t)\| \, dt \right| = d_\Lambda(y_n, y_k)$$

This allows the construction of the required Cauchy sequence (u_n) in O^+M by defining

$$u_n = \lim_{\nu\to\infty} c_\nu^*(t_n)$$

which exists by virtue of the completeness (II.2.3) of each fibre $\Pi_{\bar{O}^+}^{\leftarrow}(\Pi_U(y_n))$. The Cauchy property follows for (u_n) from that for (y_n) because for all $n, k \in \mathbb{N}$

$$d(u_n, u_k) = d_\Lambda(y_n, y_k) \quad \blacksquare$$

Corollary 1. For all $u \in \bar{O}^+M$ and all $b \in \bar{U}^+M$ there exists $v \in \Pi_{\bar{O}U}{}^{\leftarrow}(b)$ such that

$$\bar{d}(u, v) = \bar{d}_\Lambda(\Pi_{\bar{O}U}(u), b)$$

Proof. This is simply an extension of our construction in the foregoing result to the boundary. It obtains by the compactness of the orthogonal group and the transitivity of the action of $\hat{O}^+$ on the fibres of O^+M (see 1.1, II.2.5). $\blacksquare$

Corollary 2. For all $b \in \bar{U}^+M$ and all $u, v \in \Pi_{\bar{O}U}{}^{\leftarrow}(b)$ we have $\Pi_{\bar{O}}(u) = \Pi_{\bar{O}}(v) \in \bar{M} = \bar{O}^+M/\hat{O}^+$.

Proof. We lift a Cauchy sequence defining b to Cauchy sequences (u_n) and (v_n) defining u and v. Again, transitivity of $\hat{O}^+$ yields a sequence (h_n) in $\hat{O}^+$ mapping one into the other:

$$R_{h_n}(u_n) = v_n, \qquad \text{for all } n \in \mathbb{N}$$

By compactness there will exist some limit point $h \in \hat{O}^+$; then $\bar{R}_h$ links u and v in the same fibre of $\bar{O}^+M$. $\blacksquare$

2.5. The Sachs Homeomorphism $\bar{M} = \bar{O}^+M/\hat{O}^+ \simeq \bar{U}^+M/\approx$. What we have been preparing for is the partition of $\bar{U}^+M$ by some suitable equivalence relation $\approx$ so that the space of classes $\bar{U}^+M/\approx$ is topologically equivalent to $\bar{M}$. Plainly, we must have

$$y \approx z \Leftrightarrow \Pi_{\bar{U}}(y) = \Pi_{\bar{U}}(z)$$

where, using Corollary 2, we define the projection $\Pi_{\bar{U}}$ as

$$\Pi_{\bar{U}}: \bar{U}^+M \to \bar{M}: b \mapsto \Pi_{\bar{O}}(\Pi_{\bar{O}U}^{\leftarrow}(b))$$

Necessary as this is, it is rather untidy mathematics actually to *define* $\approx$ by this requirement. For it involves the simultaneous use of the Sachs metric and the Schmidt metric on their respective spaces. In fact Sachs [52] was able to formulate $\approx$, and therefore $\Pi_{\bar{U}}$, wholly in terms of his space $(U^+M, \hat{\mathbf{g}})$.

Definition (Sachs [52]). For all $y, z \in \bar{U}^+M$ we write $y \approx z$ if and only if there exist Cauchy sequences (y_n) and (z_n) in U^+M such that

(i) $\lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty} z_n = z$;
(ii) $\Pi_U(y_n) = \Pi_U(z_n)$ for all $n \in \mathbb{N}$;
(iii) there exists a uniform lower bound $A \in (-\infty, -1)$ such that for all $n \in \mathbb{N}$ (see 2.1)

$$\mathbf{g}(\iota(y_n), \iota(z_n)) \geqslant A$$

Proposition. For all $y, z \in \bar{U}^+M$,

$$y \approx z \Leftrightarrow \Pi_{\bar{U}}(y) = \Pi_{\bar{U}}(z)$$

Also, $\approx$ is an equivalence relation and so the space $\bar{U}^+M/\approx$ is homeomorphic to $\bar{M}$.

Proof (see Sachs [52]). (a) Suppose that $\Pi_{\bar{U}}(y) = \Pi_{\bar{U}}(z)$. Then there exist Cauchy sequences (u_n), (v_n) in O^+M such that

$$\lim_{n\to\infty} u_n = u \in \Pi_{\bar{O}U}^{\leftarrow}(y)$$

$$\lim_{n\to\infty} v_n = v \in \Pi_{\bar{O}U}^{\leftarrow}(z)$$

$$\lim_{n\to\infty} R_h(u_n) = v, \qquad \text{for some } h \in \hat{O}^+$$

Now construct a sequence (y_n) by alternating terms from $\Pi_{OU}(u_n)$ and $\Pi_{OU} \circ R_h^{-1}(v_n)$ and, similarly, a sequence (z_n) by alternating terms from $\Pi_{OU} \circ R_h(u_n)$ and $\Pi_{OU}(v_n)$. These are convergent to y and z respectively. Moreover, since h is fixed for all $n \in \mathbb{N}$ the required uniform bound obtains. Therefore $y \approx z$.

(b) Suppose that $y \approx z$. Then there exist sequences (y_n) and (z_n) in U^+M with a common projection and convergent to y and z respectively. By Corollary 1 in 2.4 we can lift (y_n) and (z_n) to Cauchy sequences (u_n) and (v_n) in O^+M. We obtain a sequence $(h_n) \in \hat{O}^+$ by defining

$$R_{h_n}(u_h) = v_n, \quad \text{for all } n \in \mathbb{N}$$

By hypothesis, there is a uniform lower bound on the expression $\mathbf{g}(\iota(y_n), \iota(z_n)$ so (h_n) has a cluster point $h \in \hat{O}^+$. Then

$$\lim_{n\to\infty} R_h(u_n) = \lim_{n\to\infty} v_n$$

and so $\Pi_{\bar{U}}(y) = \Pi_{\bar{U}}(z)$.

(c) It remains to observe that property (iii) of $\approx$ is transitive, though it is not obviously so. We recall that a relation is an equivalence relation if and only if it partitions the set into disjoint nonempty subsets. That is precisely what $\Pi_{\bar{U}}$ does, and the fibres it determines are just the classes of $\approx$. So, as required, $\bar{U}^+M/\approx$ is homeomorphic to $\bar{M}$. ■

2.6. Completion of Bundles Associated with *LM*. We are particularly interested for space–time in the associated bundles $(LM \times F)/G$, where F is $\mathbb{R}^n$ or a tensor product constructed therefrom (see I.4.3 and Examples 2, 3). These are the tensor bundles $T_h{}^kM$. Among the sections of them we find the Riemann tensor, the energy–momentum tensor, and the Lorentz metric. All have considerable physical significance and so it is important to investigate their behavior as the boundary ∂M is approached. This is made possible by Hájiček and Schmidt [32] who effected the completion of all bundles associated with LM. We adapt the construction given by these authors to suit our previous results and notation.

Let $L'M$ be a connected component of LM and suppose that $(L'M \times F)/G^+$ is any associated bundle (see I.4.3). Then the right action of G^+ on $L'M \times F$ is given by

$$(L'M \times F) \times G^+ \to (L'M \times F): (u, a, h) \mapsto (R_h(u), L_{h^{-1}}(u))$$

and we have the smooth, open projection

$$\tilde{\Pi}_{L'}: (L'M \times F)/G^+ \to M: R_{G^+}(u, a) \mapsto \Pi_{L'}(u)$$

From the completion of $L'M$ (see II.1.4) we have the continuous, open projection

$$\Pi_{\bar{L}'}: \bar{L}'M \to \bar{M}: u \mapsto \bar{R}_{G^+}(u)$$

We fit these together to obtain the following spaces and continuous, open projections:

$$\Pi_{\bar{1}}: \bar{L}'M \times F \to \bar{L}'M: (u, a) \mapsto u$$

$$\Pi_{\bar{F}}: \bar{L}'M \times F \to (\bar{L}'M \times F)/G^+: (u, a) \mapsto \bar{R}_{G^+}(u, a)$$

$$\tilde{\Pi}_{\bar{L}'}: (\bar{L}'M \times F)/G^+ \to \bar{M}: \bar{R}_{G^+}(u, a) \to \Pi_{\bar{L}'}(u)$$

They form the following commutative diagram:

$$\begin{array}{ccc} \bar{L}'M \times F & \xrightarrow{\Pi_{\bar{1}}} & \bar{L}'M \\ {\scriptstyle \Pi_{\bar{F}}} \downarrow & & \downarrow {\scriptstyle \Pi_{\bar{L}'}} \\ (\bar{L}'M \times F)/G^+ & \xrightarrow[\tilde{\Pi}_{\bar{L}'}]{} & \bar{M} \end{array}$$

For any open neighborhood $A \subseteq M$ of $x \in M$ the associated bundle property I.4.3(ii) assures us that $\tilde{\Pi}_{\bar{L}'}{}^{\leftarrow}(A) = \tilde{\Pi}_{\bar{L}'}{}^{\leftarrow}(A)$ is diffeomorphic to $A \times F$. In particular, for a vector space F, there would exist a linear isomorphism with each fibre $\tilde{\Pi}_{\bar{L}'}{}^{\leftarrow}(y)$ for $y \in A$. However, this need not be the case for a point $z \in \partial M = \bar{M} \backslash M$ the b-boundary of M. What matters is whether or not G^+ acts freely on the fibre $\tilde{\Pi}_{\bar{L}'}{}^{\leftarrow}(z)$ (see I.3.8). Equivalently, we need to know whether the subgroup

$$G_u{}^+ = \{h \in G^+ | \bar{R}_h(u) = u\}$$

is trivial, for any $u \in \tilde{\Pi}_{\bar{L}'}{}^{\leftarrow}(z)$. We denote by $F/G_u{}^+$ the quotients pace with the topology that makes continuous and open the canonical projection

$$F \to F/G_u{}^+ : a \mapsto [a] = L_{G_u^+}(a)$$

Now everything falls into place by means of the following results of Hájiček and Schmidt [32].

Proposition. Let $z \in \bar{M}$, $u \in \Pi_{\bar{L}'}{}^{\leftarrow}(z)$. Then there is a homeomorphism $\tilde{\Pi}_{\bar{L}'}{}^{\leftarrow}(z) \simeq F/G_u{}^+$. Moreover, if $G_u{}^+$ is trivial and F is a vector space, then there is a linear isomorphism $\tilde{\Pi}_{\bar{L}'}{}^{\leftarrow}(u) \cong F$.

Proof. (i) By construction we have a continuous, open map

$$f_u : F/G_u{}^+ \to \tilde{\Pi}_{\bar{L}'}{}^{\leftarrow}(z) : [a] \mapsto \Pi_{\bar{F}}(u, a)$$

We show that it is bijective and therefore a homeomorphism.

Suppose $(u, b) \in \tilde{\Pi}_{\bar{L}'}{}^{\leftarrow}(z)$. Then there exists some $h \in G^+$ such that $\bar{R}_h(u, b) = (u, a)$. This means that

$$(\bar{R}_h(u), L_{h^{-1}}(b)) = (u, a)$$

So $h \in G_u{}^+$, $L_{h^{-1}}(b) = a$ and hence $b \in [a]$, which implies that $[b] = [a] \in F/G_u{}^+$.

Now suppose $f_u[a] = f_u[b]$ for some $a, b \in F$. Then

$$\bar{R}_{G^+}(u, a) = \bar{R}_{G^+}(u, b)$$

so for some $h \in G_u{}^+$, $L_{h^{-1}}(a) = b$ and $[a] = [b]$.

(ii) Let F be a vector space and let $G_u{}^+$ be trivial. Then, from (i), $\tilde{\Pi}_{\bar{L}'}{}^{\leftarrow}(z) \simeq F/G_u{}^+$ and by hypothesis $E/G_u{}^+ \cong F$, since $G_u{}^+$ is trivial. We have only to use f_u to induce a well-defined vector space structure on $\tilde{\Pi}_{L'}^{\leftarrow}(z)$. The standard procedure is to define for all $X, Y \in \tilde{\Pi}_{L'}^{\leftarrow}(z)$ and $\lambda \in \mathbb{R}$:

$$X + Y = f_u(f_u^{-1}(X) + f_u^{-1}(Y))$$
$$\lambda X = f_u(\lambda f_u^{-1}(X))$$

This necessarily makes f_u linear, and since it is bijective it is an isomorphism. It remains to show that the whole construction is independent of the particular choice of u. Suppose then that we also have $v \in \Pi_{L'}^{\leftarrow}(z)$. It follows that since G_u^+ is trivial so also is G_v^+. For suppose $\bar{R}_k(v) = v$. Then by transitivity there exists $m \in G^+$ with $\bar{R}_m(u) = v$ and so we find

$$\bar{R}_k(v) = \bar{R}_k \circ \bar{R}_m(u) = \bar{R}_m(u)$$

But since G_u^+ consists only of the identity e, we have

$$\bar{R}_{m^{-1}} \circ \bar{R}_k \circ \bar{R}_m = \bar{R}_e, \quad \text{so } \bar{R}_k = \bar{R}_e$$

Therefore, G_v^+ is trivial and $F/G_v^+ = F/G_u^+ = F$ since the classes are singletons. Hence:

$$f_u: F \to \tilde{\Pi}_{L'}^{\leftarrow}(z): a \mapsto \bar{R}_{G^+}(u, a)$$
$$f_v: F \to \tilde{\Pi}_{L'}^{\leftarrow}(z): b \mapsto \bar{R}_{G^+}(v, b)$$

Let $X = \bar{R}_{G^+}(u, a)$ and $Y = \bar{R}_{G^+}(v, b)$. We find $X + Y$ via f_u and via f_v as follows, with $\bar{R}_m(u) = v$ as before,

$$f_u(f_u^{-1}(X) + f_u^{-1}(Y)) = f_u(a + L_m b) = \bar{R}_{G^+}(u, a + L_m b)$$
$$f_v(f_v^{-1}(X) + f_v^{-1}(Y)) = f_v(L_{m^{-1}} a + b) = \bar{R}_{G^+}(v, L_{m^{-1}} a + b)$$

But these are the same because

$$\bar{R}_{G^+}(u, a + L_m b) = \bar{R}_{G^+}(\bar{R}_m(u, a + L_m b)) = \bar{R}_{G^+}(\bar{R}_m(u), L_{m^{-1}}(a + L_m b))$$

A similar argument holds good for multiplication by scalars, so the vector space structure is well defined for $\Pi_{L'}^{\leftarrow}(z)$ and it is isomorphic to F. ■

Remarks. (1) We can take the particular case $F = \mathbb{R}^4$ and obtain the completion space $\overline{TM}$ in which the tangent bundle TM is dense. Then the Lorentz metric for space–time $(M, \mathbf{g})$ admits by I.1.5) a continuous extension $\bar{\mathbf{g}}: \overline{TM} \times \overline{TM} \to \mathbb{R}$, which agrees with $\mathbf{g}$ on $TM \times TM$. Of course, if $z \in \partial M$ is such that $\Pi_{\bar{T}}^{\leftarrow}(z)$ is *degenerate* (not a vector space isomorphic to $\mathbb{R}^4$), then we do not expect all the algebraic properties of $\mathbf{g}$ to extend to $\bar{\mathbf{g}}_z$ at z.

On the other hand, if $z \in \partial M$ is such that $\Pi_{\bar{T}}^{\leftarrow}(z) \cong \mathbb{R}^4$, then there is a well-defined "tangent space" $T_z\bar{M}$ at z. In this case the symmetry, bilinearity, nondegeneracy, and signature that characterize $\mathbf{g}$ ought to persist in its extension to $\bar{g}_z$. Indeed we can always find some x in M and a b-incomplete curve (see II.3.9)

$$c\colon [0, 1) \to M, \quad \text{with endpoint } z \in \partial M$$

Parallel transport along c will generate for all $t \in [0, 1)$ an isomorphism (see I.5.4)

$$\tau_t\colon \Pi_{\bar{T}}^{+}(x) \to \Pi_{\bar{T}}^{+} \circ c(t)$$

which is an isometry. By continuity this extends to $t = 1$ as the isometry

$$\tau_1\colon \Pi_{\bar{T}}^{+}(x) \to \Pi_{\bar{T}}^{+}(z)\colon u \mapsto \lim_{t \to 1} \tau_t u$$

So we have for all $v, w \in T_z\overline{M}$

$$\bar{g}_z(v, w) = g_x(\tau_t^{-1}v, \tau_t^{-1}w)$$

which preserves the Lorentz structure of $g_{c(t)}$ in its extension to $\bar{g}_z = \lim_{t\to 1} g_{c(t)}$. A more sophisticated detailed proof is given in Hájiček and Schmidt [32].

(2) Duncan and Shepley [19] have pointed out that

$$F = F_T = \{(x^i) \in \mathbb{R}^4 | (x^1)^2 = 1 + (x^2)^2 + (x^3)^2 + (x^4)^2\}$$

generates the unit future timelike bundle U^+M used by Sachs [52] (cf. 2.1). Let $\hat{U}^+$ denote the subgroup of $\hat{O}^+$ that preserves time orientation; then we have

$$U^+M = (L'M \times F_T)/\hat{U}^+$$

which by the foregoing procedure is dense in the topological space $(\bar{L}'M \times F_T)/\hat{U}^+ = \tilde{U}^+M$.

Note that Duncan and Shepley [19], pp. 488, 490, misquote the equivalence relation of Sachs by omitting condition (iii) in 2.5. However, they observe that the relation $\approx$ on $\bar{U}^+M$ could be replaced by $\dot{\approx}$ on $\bar{U}^+M$, where (see definition of $\approx$ in 2.5) $y \dot{\approx} z$ if and only if there exist Cauchy sequences (y_n), (z_n) in the metric space (U^+M, d_Λ) such that there exists

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n$$

in the topological space $\tilde{U}^+M$. Our description here emphasizes the involvement of two topologies in the application of $\dot{\approx}$. Duncan and Shepley prefer it because $\dot{\approx}$ avoids directly using information about sequences on M, but they did not comment on its mixed heritage. Certainly the factorization $\bar{U}^+M/\dot{\approx}$ more readily admits a homeomorphism with $\overline{M}$ than was the case for $\bar{U}^+M/\approx$ in 2.5.

(3) In another article, Duncan and Shepley [18] suggest that for any point $z \in \partial M$, a coarse method of classifying the singularity there is via the

difference between the dimension of M and the "dimension of $\Pi_{\bar{T}}^{-}(z)$." Their argument appears to suppose that $\Pi_{\bar{T}}^{-}(z)$ will always be a vector space. Furthermore, elsewhere in the article, they assign a global dimension to ∂M and assume that it coincides with the dimension of $\Pi_{\bar{T}}^{-}(z)$ for all z. What can be salvaged from their argument is that if M is extensible to a differentiable manifold containing $\bar{M}$, then we must have $\Pi_{\bar{T}}^{-}(z)$ isomorphic to $\mathbb{R}^4$ for all $z \in \partial M$, unless we are prepared to use a more general definition for dimension (as, for example, in Hurewicz and Wallman [36], p. 4).

2.7. Boundary Points Generated by Horizontal Cauchy Sequences. Every $\bar{x}$ in the b-boundary ∂M is the projection by $\Pi_{\bar{O}^+}$ of an equivalence class of Cauchy sequences on O^+M. If the Cauchy sequence (v_n) determines $\bar{x} \in \partial M$, then this $\bar{x}$ is equivalently determined by a Cauchy sequence (u_n) on a horizontal curve in O^+M.

Proof. This result was obtained by Schmidt [55] through a fibre isometry with $L'\mathbb{R}^n$ which is a complete Riemannian manifold. We give a direct proof of similar length.

There is a curve $k\colon [0, 1) \to O^+M$ such that, for some sequence $(t_n) \subset [0, 1)$ convergent to 1, $v_n = k(t_n)$. Now $k(t)$ does not eventually lie wholly in one fibre $\Pi_{\bar{O}^+}^{-}(z)$, for then by 2.3 (v_n) would have a limit there instead of in the boundary $\bar{O}^+M \backslash O^+M$. Hence $\Pi_{O^+} \circ k = c\colon [0, 1) \to M$ is inextensible in M. We take the unique horizontal lift (see I.5.4)

$$c^{\uparrow}\colon [0, 1) \to O^+M\colon t \mapsto (c(t), \tau_t(b_1)), \quad \text{with } c^{\uparrow}(0) = v_1 = (x_1, b_1)$$

Now we have a sequence $(c^{\uparrow}(t_n))$ in O^+M with projection $(x_n) = (\Pi_{O^+}(v_n))$ in M. The action of $\hat{O}^+$ is by 2.1 transitive on fibres, so for all $n \in \mathbb{N}$ we can find

$$g_n \in \hat{O}^+ : R_{g_n}(c^{\uparrow}(t_n)) = v_n$$

From property I.5.1(ii) of a connection we know also that elements of $\hat{O}^+$ will map a horizontal curve into another horizontal curve. $\hat{O}^+$ has the standard topology induced by $\mathbb{R}^{16}$, it is closed in $Gl(4; \mathbb{R})$, by Schmidt [54]; cf. Friedrich [25], p. 690.

Suppose that v_∞ is the limit of (v_n); it is known to be in $\bar{O}^+M/O^+M$. By completeness of $\bar{O}^+M$ the curve $c^{\uparrow}$ has an endpoint $u_\infty \in \bar{O}^+M \backslash O^+M$ with

$$\Pi_{\bar{O}^+}(u_\infty) = \Pi_{\bar{O}^+}(v_\infty) = \bar{x} \in \partial M$$

Hence there exists some $g_\infty \in \hat{O}^+$ with $R_{g_\infty}(u_\infty) = v_\infty$. By continuity the sequence (g_n) defined above converges to g_∞. Let c^* be the unique horizontal lift of c through $R_{g_\infty}(v_1)$, that is, $c^*(0) = R_{g_\infty}(v_1)$. Recall that $\bar{R}_{g_\infty}$ coincides with R_{g_∞} on O^+M.

We define the sequence $(c^*(t_n)) = (u_n)$ on the horizontal curve c^*. Now, parallel transport commutes with the action of $\hat{O}^+$ (see I.5.4) and so $c^* = R_{g_\infty} \circ c^\uparrow$. Therefore for all $n \in \mathbb{N}$, $u_n = R_{g_\infty}(c^\uparrow(t_n))$. We find that (u_n) is Cauchy with the same limit $v_\infty \in \bar{O}^+M \backslash O^+M$ as (v_n), because

$$\begin{aligned} d(u_n, u_h) &\leqslant d(u_n, v_n) + d(v_n, v_k) + d(v_k, u_k) \\ &\leqslant d(R_{g_\infty} \circ c^\uparrow(t_n), R_{g_n} \circ c^\uparrow(t_n)) + d(v_n, v_k) \\ &\quad + d(R_{g_k} \circ c^\uparrow(t_k), R_{g_\infty} \circ c^\uparrow(t_k)) \end{aligned}$$

Now (v_n) is Cauchy, and $(g_n) \subset \hat{O}^+$ is a sequence of continuous maps with limit $g_\infty \in \hat{O}^+$. So for all $\epsilon > 0$ we can find $N_\varepsilon \in \mathbb{N}$ such that

$$d(u_n, u_k) < \epsilon \quad \text{for } n, k > N_\varepsilon \quad \blacksquare$$

Example. Consider L^+S^1 with constant connection $\lambda \neq 0$. We have $L^+S^1 = \{(x, b) \in S^1 \times \mathbb{R}^+\}$, and a Cauchy sequence in L^+S^1 is given, for example, by $(u_n): n \mapsto (a, e^{\lambda n})$, for some fixed $a \in S^1$. Plainly this has no limit in L^+S^1. It is Cauchy because for any $n, k \in \mathbb{N}$ the points u_n, u_k lie on the curve

$$c: t \mapsto (a - t, e^{\lambda t})$$

with tangent vector field, by 1.1

$$\dot{c}: t \mapsto (-1, \lambda e^{\lambda t}) \quad \text{and} \quad \|\dot{c}(t)\| = e^{-\lambda t}$$

Therefore by I.2.4

$$d(u_n, u_k) \leqslant \left| \int_n^k e^{-\lambda t}\, dt \right| = \frac{1}{|\lambda|} \left|e^{-\lambda k} - e^{-\lambda n}\right|$$

In fact by I.5.4 the curve c is a horizontal curve: the horizontal lift of $\mathbb{R} \to S^1: t \mapsto a - t$ through $1 \in \mathbb{R}^*$ at $t = 0$.

Note that in the uniformly equivalent metric space [17]

$$\{(u, v) \in S^1 \times (1/|\lambda|, \infty)\} \cong L^+S^1$$

our horizontal curve appears as (see Example 2 in II.1.4)

$$\begin{aligned} u(t) &= \text{constant} = \lambda(a - t) + \log e^{\lambda t} = \lambda a \\ v(t) &= (e^{-\lambda t} + 1)/|\lambda| \end{aligned}$$

Also, our Cauchy sequence becomes

$$n \mapsto (\lambda a, (e^{-\lambda n} + 1)/|\lambda|)$$

and we can easily see that it has the limit

$$(\lambda a, 1/|\lambda|) \in S^1 \times [1/|\lambda|, \infty) \cong \bar{L}^+S^1$$

with respect to the usual metric.

2.8. Holonomy Bundles Generate the Same b-Boundary. The holonomy bundle through any point in the frame bundle determines the same b-boundary as the frame bundle itself.

Proof. Schmidt [55] pointed out the equivalence of this result with our 2.7. We have encountered the generalization due to Friedrich [25] in II.2.5.

Let $u_0 \in O^+M$ be arbitrary, and denote the holonomy bundle through u_0 (see I.5.8) by

$$L'M(u_0) = \{v \in L'M \,|\, u_0 \sim v\}$$

It inherits a metric from inclusion in the Riemannian manifold $(O^+M, \langle\ ,\ \rangle)$. From 2.7, for all $\bar{x} \in \partial M$ we can find a Cauchy sequence (u_n) on a horizontal curve in O^+M with

$$\Pi_{\bar{O}^+} \lim (u_n) = \bar{x}$$

So we can take the Cauchy Completion $\bar{L}'M(u_0)$ and extend the action of $\Phi(u_0)$ uniformly continuously as before and find

$$\Pi_{\bar{L}'}(\bar{L}'M(u_0)) = \bar{L}'M(u_0)/\Phi(u_0) \simeq \bar{M} = M \cup \partial M \quad \blacksquare$$

Remark. Ihrig [37] has found the holonomy groups of a large class of space–times.

Example. L^+S^1 with constant connection λ, again! (Cf. Example 2 of I.5.7.) Let $u_0 = (x_0, b_0) \in L^+S^1$ be arbitrary; then we have

$$\Phi(u_0) = \{e^{-\lambda k} \in \mathbb{R}^+ \,|\, k \in \mathbb{Z}\}$$

$$L^+S^1(u_0) = \{(x(\text{mod } 1), b_0 \exp\{-\lambda(x - x_0)\}) \,|\, x \in \mathbb{R}\}$$

From Example II.1.3 we have the isometric equivalence

$$L^+S^1(u_0) \cong \{((\lambda x_0 + \log b_0) \bmod \lambda, (\exp\{\lambda(x - x_0)\}/b_0|\lambda|) + 1/|\lambda|) \,|\, x \in \mathbb{R}\}$$

This allows us more easily to see the completion

$$\bar{L}^+S^1(u_0) \cong \{((\lambda x_0 + \log b_0) \bmod \lambda, v) \,|\, v \geqslant 1/|\lambda|\}$$

The action of $\Phi(u_0)$ was given implicitly in II.1.3; it easily extends to

$$\begin{aligned} &\bar{R}_{e^{-\lambda k}}((\lambda x_0 + \log b_0) \bmod \lambda, v) \\ &\qquad = ((\lambda x_0 + \log b_0 - \lambda k) \bmod \lambda, (v - 1/|\lambda|)e^{\lambda k} + 1/|\lambda|) \\ &\qquad = ((\lambda x_0 + \log b_0) \bmod \lambda, (v - 1/|\lambda|)e^{\lambda k} + 1/|\lambda|) \end{aligned}$$

Hence we find the quotient (see I.3.9) from

$$\begin{aligned} &[((\lambda x_0 + \log b_0) \bmod \lambda, v)] \\ &\qquad = \{((\lambda x_0 + \log b_0) \bmod \lambda, (v - 1/|\lambda|)e^{\lambda k} + 1/|\lambda|) \,|\, k \in \mathbb{Z}\} \\ &\bar{L}^+S^1(u_0)/\Phi(u_0) \cong \{[((\lambda x_0 + \log b_0) \bmod \lambda, v)] \,|\, v \geqslant 1/|\lambda|\} \end{aligned}$$

As required this agrees with (see Example 2 of II.1.4)

$$\bar{L}^+S^1/R^+ \simeq S^1 \cup \{\Omega\}$$

for the point Ω corresponds to the singleton class

$$[((\lambda x_0 + \log b_0) \bmod \lambda, 1/|\lambda|)]$$

and as v runs through the set $(1/|\lambda|, \infty)$ so the corresponding point $(x, b) \in L^+S^1$ runs through values given by (cf. Example 1.3)

$$\lambda x + \log b = \lambda x_0 + \log b_0$$
$$b = 1/(v|\lambda| - 1)$$

Hence x runs through $\{\lambda x_0 + \log b_0 + \log(v|\lambda| - 1)|v > 1/|\lambda|\}$, which means that modulo 1 it certainly completes a circuit of S^1.

2.9. Approaching the b-Boundary. We are now in a fairly strong position to investigate the geometry of space–time $(M, \mathbf{g})$ near the b-boundary ∂M. We have topologies for all the following homeomorphs of $\bar{M} = M \cup \partial M$:

$\bar{L}'M/G^+$	(cf. II.1.4)
$\bar{L}'M(u)/\Phi(u)$	(cf. II.2.6)
$\bar{O}^+M/\hat{O}^+$	(cf. II.2.5)
$\bar{U}^+M/\approx$	(cf. III.2.5)
$\bar{U}^+M/\dot{\approx}$	(cf. III.2.6)

Thus we are well equipped indeed to say what we mean by "near the b-boundary."

In the previous section we obtained from $\bar{L}'M$ the completion of the tensor bundles and discovered the topology of the limiting, tangent tensor spaces at the b-boundary. For nondegenerate such spaces we could reconstruct the appropriate algebraic properties. Hence, we can investigate the vicinity of the b-boundary by studying there the limits of geometrically significant tensor fields. We shall return to this theme again later (4.2), but here we give some preliminaries.

Proposition. Let $c\colon [0, 1) \to M$ be a b-incomplete curve with endpoint $z \in \partial M$.

Let $c^*\colon [0, 1) \to L'M$ be any continuous curve with $\Pi_{L'} \circ c^* = c$ and endpoint $u \in \Pi_{\bar{L}'}^{\leftarrow}(z)$.

Let $a\colon [0, 1) \to F$ be a curve (in the manifold F on which G^+ acts on the left) such that for some $a' \in F$ all neighborhoods N of $L_{G_u^+}(a')$ have a real $\delta > 0$ for which $a(t) \in N$ for $t > (1 - \delta)$.

Then the field

$$X: [0, 1) \to (L'M \times F)/G^+ : t \mapsto \Pi_F(c^*(t), a(t))$$

has a well-defined limit

$$\lim_{t \to 1} X(t) = \Pi_{\bar{F}}(u, a') \in (\bar{L}'M \times F)/G^+$$

Proof. (See Hájiček and Schmidt [32], though beware the false economy in notation!) From 2.6 we recall that

$$G_u{}^+ = \{h \in G^+ | \bar{R}_h(u) = u\}$$

Let V be any set containing $\{X(t)|0 < t < 1\}$ which is open in $(\bar{L}'M \times F)/G^+$. By the continuity of

$$\Pi_{\bar{F}}: \bar{L}'M \times F \to (\bar{L}'M \times F)/G^+$$

we have that $\Pi_{\bar{F}}^{\leftarrow}(V)$ is a neighborhood of

$$\{u\} \times \bar{L}_{G_u^+}(a') \subseteq \bar{L}'M \times F$$

By hypothesis there exist real $\delta > 0$ such that

$$(c^*(t), a(t)) \in \Pi_{\bar{F}}^{\leftarrow}(V), \quad \text{for } t > 1 - \delta$$

Therefore the projection of this curve,

$$\{X(t)|t > 1 - \delta\}$$

lies in V and so has a limit in V^c. Again, by continuity of $\Pi_{\bar{F}}$ this limit coincides, as required, with $\Pi_{\bar{F}}(u, a')$. ■

Remarks. (1) Hájiček and Schmidt pointed out that in particular this result implies the following. Suppose that

$$X: [0, 1) \to (L'M \times F)/G^+$$

is a tensor field, along a curve c: $[0, 1) \to M$, which terminates at $z \in \partial M$, and that the tangent space at $z \in M$ is nondegenerate. Then a limiting value for X at z exists if its components (X^i) in a parallel propagated frame along c, given by

$$c^*: [0, 1) \to L'M: t \mapsto \tau_t(b_i)$$

have well-defined limits as $t \to 1$. Conversely, such tensor fields can be constructed from boundary values by taking the (X^i) constant along the chosen curve.

Hájiček and Schmidt also formulated the notion of accessibility for a boundary point $z \in \partial M$.

Let $c: (0, 1) \to M$ be a smooth curve with endpoint $z \in \partial M$. So

$\bar{c}\colon (0, 1] \to \bar{M}$ is continuous with $\bar{c}(1) = z$. We say that $\bar{c}$ has a tangent vector $X \in T_z\bar{M}$ if there is a continuous map

$$\dot{\bar{c}}\colon (0, 1] \to TM, \quad \text{with} \quad \dot{\bar{c}}(t) = \dot{c}(t) \quad \text{for } t < 1$$

such that $X = \dot{\bar{c}}(1)$. The *accessibility* of $z \in \partial M$ is the set of elements

$$A_z = \{X \in T_z\bar{M} \mid \exists \bar{c} \text{ with tangent vector } X\}$$

with $\bar{c}$ as in the previous construction. Hájiček and Schmidt illustrated the notion for a flat two-dimensional space–time $M \subset \mathbb{R}^2$ where the topological boundary $\dot{M}$ (see I.1.12) of M in $\mathbb{R}^2$ is a piecewise continuously differentiable curve c. Then the b-boundary ∂M coincides with c, and the boundary of $L'M$ in $\bar{L}'M$ is $\Pi_{\bar{L}'}^{\leftarrow}(c)$ with $\bar{T}M\backslash TM = \Pi_{\bar{T}}^{\leftarrow}(c)$.

(2) Duncan and Shepley [19] showed how some information could be gained about a b-boundary by studying the algebraic structure of the bundle metric. Thus, if the latter has a zero eigenvalue somewhere, then the integral curves of the corresponding eigenvectors yield the b-boundary through identification by the appropriate structure group. Of course a zero eigenvalue corresponds to a vanishing determinant for the metric tensor in local coordinates. Geometrically that corresponds to a collapse of distances in one or more directions and hence an identification of some superficially distinct Cauchy sequences in the bundle.

(3) Clarke [10] has shown that in the situations envisaged by the singularity theorems of Hawking and Penrose, the Riemann tensor cannot be well behaved at points of the b-boundary. The proof requires that $(M, \mathbf{g})$ be a globally hyperbolic space–time, so $M \simeq \mathbb{R} \times S$ for some three-manifold S (see [35], p. 212), but $\mathbf{g}$ need only be of class C^{1-} (see 1.2 and 4.2). For the pseudo-Riemannian cylinder N studied in I.5.9,10 we see that singularity of the metric arises as $\psi \to 0$, and there one component of the Riemann tensor is

$$R^1_{212}(\psi) \to \psi^{-2}, \quad \text{as } \psi \to 0$$

3. FRIEDMANN SPACE–TIMES

Observations of our own universe indicate a high degree of spherical symmetry about our own position, both in the disposition of luminous matter and in the background blackbody radiation. Since we have no reason to suppose that our position is in any way special, it is natural to consider cosmological models that display spherical symmetry about every point and are therefore spatially homogeneous. There exist exact solutions of Einstein's equation (see 1.3) for such conditions, and they are called Friedmann or Robertson–Walker space–times (see [35], pp. 134–142). We shall consider such a space–time $(M, \mathbf{g})$ with metric $\mathbf{g}$ given in coordinates by the arc length formula

$$ds_{\mathbf{g}}^2 = R^2(\psi)(-d\psi^2 + d\sigma^2 + \sin^2\sigma(d\theta^2 + \sin^2\theta\, d\varphi^2))$$

Here ψ is the time coordinate and σ, θ, φ are polar angles on $S^3 \subset \mathbb{R}^4$. The smooth, positive function R is defined for ψ in some interval $(0, T) \subset \mathbb{R}$. The physical requirements of positive matter density, nonnegative pressure, and the observed recession of galaxies leads to the behavior $R(\psi) \to 0$ as $\psi \to 0$. Without loss of generality we can suppose that $R(\psi)$ behaves like ψ^ρ for some $\rho > 0$, as $\psi \to 0$; in the particular case of Example 2, I.5.9, $\rho = 2$. It follows that the model indicates a physical singularity "at $\psi = 0$" because the matter density increases without bound as $\psi \to 0$. This singularity can be incorporated into the b-boundary

$$\partial M = \bar{O}^+M/\hat{O}^+\backslash M$$

Bosshard [7] showed that (i) fibres of $\bar{O}^+M$ over $\psi = 0$ are degenerate; (ii) if also $R(\psi) \to 0$ as $\psi \to T$ for some $T \in \mathbb{R}$, then there is a similar singularity "at $\psi = T$" and the two singularities are identified in ∂M if $R(\psi) \sim R(T - \psi)$ as $\psi \to 0$. Independently of Bosshard [7], Johnson [38] used similar techniques to extend (i) to a wider class of space–times (including that of Schwarzschild) and to prove that in all such cases the bundle completion $\bar{M} = M \cup \partial M$ is non-Hausdorff.

We give some details of the calculations that lead to these results. After considering the implications we introduce in 3.8 a new bundle metric for parallelizable manifolds (see I.2.7 and the opening remarks of this part). This shows promise of some advantages for the completion of space–times; Clarke has derived a two-stage completion procedure as an alternative, not requiring a parallelization (see 4.4) and showing even more promise.

3.1. The Injection $h\colon O^+N \to O^+M$. In the metric just given we can of course ignore the coordinate singularities when $\sin\sigma$ or $\sin\theta$ is zero because these are irrelevant to the geometry and physics. On the other hand, the behavior $R(\psi) \to \psi^\rho$ as $\psi \to 0$ is not coordinate dependent and it heralds a true geometric singularity. Bosshard [7] pointed out that it is sufficient to consider the two-manifold $(N, \boldsymbol{\gamma})$ where

$$N = \{(\psi, \sigma) | \psi \in (0, T), \sigma \in S^1\}$$

and the pseudo-Riemannian metric $\boldsymbol{\gamma}$ has line element

$$ds_{\boldsymbol{\gamma}}{}^2 = R^2(\psi)(-d\psi^2 + d\sigma^2)$$

We shall devote some space to establish this sufficiency. The investigation of a space–time through submanifolds that display singularities can save considerable effort because the dimension of the frame bundle increases even faster than the square of the dimension of the underlying manifold.

There is an S^2-family of natural smooth injections of N into M:

$$i_1 : N \to M : (\psi, \sigma) \mapsto (\psi, \sigma, \theta_0, \varphi_0)$$

for any fixed $\theta_0, \varphi_0 \in S^2$.

The positively oriented component O^+N of the orthonormal frame bundle of N (see I.5.10) is

$$O^+N = \left\{ \left(\psi, \sigma, \begin{bmatrix} \cosh \chi & \sinh \chi \\ \sinh \chi & \cosh \chi \end{bmatrix} \frac{1}{R} \begin{bmatrix} \partial_\psi \\ \partial_\sigma \end{bmatrix} \right) \middle| (\psi, \sigma) \in N, \chi \in \mathbb{R} \right\}$$

where we have arranged to locate the orthonormal frame

$$\frac{1}{r} \begin{bmatrix} \partial_\psi \\ \partial_\sigma \end{bmatrix}$$

at $\chi = 0$ for all $(\psi, \sigma) \in N$. This is precisely what we did for the example in I.5.10, where we studied the particular case when

$$R: (0, 2\pi) \to \mathbb{R}: \psi \mapsto 1 - \cos \psi$$

Next we find, for fixed θ_0 and φ_0, a smooth injection of O^+N and O^+M. Consider the submanifold $O^+M_0 \hookrightarrow O^+M$ defined by

$$O^+M_0 = \left\{ \left((\psi, \sigma, \theta_0, \varphi_0), \begin{bmatrix} L(\chi) & 0 \\ 0 & I \end{bmatrix} \frac{1}{R} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) \middle| \chi \in \mathbb{R} \right\}$$

where

$$L(\chi) = \begin{bmatrix} \cosh \chi & \sinh \chi \\ \sinh \chi & \cosh \chi \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$b_1 = \begin{bmatrix} \partial_\psi \\ \partial_\sigma \end{bmatrix} \quad \text{and} \quad b_2 = \begin{bmatrix} (\sin \sigma)^{-1} & \partial_\theta \\ (\sin \theta \sin \sigma)^{-1} & \partial_\psi \end{bmatrix}$$

The manifolds O^+M_0 and O^+N are clearly isomorphic and we thereby have a smooth injection $h: O^+N \to O^+M$ since $(b_1, b_2)/R$ is an orthonormal frame at all points of M and O^+M_0 is a subbundle of O^+M because

$$\left\{ \begin{bmatrix} L(\chi) & 0 \\ 0 & I \end{bmatrix} \middle| \chi \in \mathbb{R} \right\}$$

is a Lie subgroup of the Lorentz structure group for O^+M. Using the natural projections, we have the following commutative diagram:

$$\begin{array}{ccc} & O^+M_0 & \\ {\scriptstyle i}\nearrow & & \nwarrow{\scriptstyle i_2} \\ O^+N & \xrightarrow{h} & O^+M \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ N & \xrightarrow{i_1} & M \end{array}$$

3.2. We give a proof of the following lemma stated by Bosshard [7].

Lemma. If c is a horizontal curve in O^+N, then $h \circ c$ is also a horizontal curve in O^+M.

Proof. In each case we use the Levi–Cività connection induced in the frame bundle by the metric tensor (cf. I.5.9). For O^+M we find the connection components as follows:

$$\Gamma_{11}^{1} = \Gamma_{22}^{1} = \Gamma_{12}^{2} = \Gamma_{13}^{3} = \Gamma_{14}^{4} = \dot{R}/R, \qquad \dot{R} = dR/d\psi$$

$$\Gamma_{33}^{2} = -\sin\sigma\cos\sigma, \qquad \Gamma_{33}^{1} = \frac{\dot{R}}{R}\sin^2\sigma, \qquad \Gamma_{44}^{1} = \frac{\dot{R}}{R}\sin^2\sigma\sin^2\theta$$

$$\Gamma_{44}^{2} = -\sin\sigma\cos\sigma\sin^2\theta, \qquad \Gamma_{23}^{3} = \Gamma_{24}^{4} = \cot\sigma$$

$$\Gamma_{44}^{3} = -\sin\theta\cos\theta, \qquad \Gamma_{34}^{4} = \cot\theta$$

By symmetry we have $\Gamma_{ij}^{k} = \Gamma_{ji}^{k}$ for all i, j, k; otherwise the remaining components are zero.

Evidently the components of the connection on O^+N are given by these Γ_{ij}^{k} with indices restricted to 1, 2. From I.5.2 a curve

$$c\colon t \mapsto (c^i(t), b_j{}^i(t))$$

is horizontal in a frame bundle if and only if its tangent vector $\dot{c}(t) = (\dot{c}^i(t), \dot{b}_j{}^i(t))$ satisfies

$$\dot{b}_j{}^i = -b_j{}^k \dot{c}^l \Gamma_{kl}^{i}, \quad \text{for all } t$$

Suppose that this curve is horizontal in O^+N. Then we obtain a curve in O^+M given by

$$h \circ c\colon t \mapsto ((c^1(t), c^2(t), \theta_0, \varphi_0), B_\beta{}^\alpha(t)), \qquad \alpha, \beta = 1, 2, 3, 4$$

where $B_j{}^i = b_j{}^i$ for $i, j = 1, 2$ and $B_3{}^3 = B_4{}^4 = 1/R$, with other components zero. We observe that $\dot{B}_j{}^i = \dot{b}_j{}^i$ for $i, j = 1, 2$ and also $\dot{c}^3 = \dot{c}^4 = 0$. Therefore for $i, j = 1, 2$ we have, as required,

$$\dot{B}_j{}^i = -B_j{}^k \dot{c}^l \Gamma_{kl}^{i}$$

It remains only to check for $B_3{}^3 = B_4{}^4$.

$$\dot{B}_3{}^3 = \dot{B}_4{}^4 = -\dot{R}\dot{c}^1/R, \quad \text{where} \quad \dot{R} = \frac{dR}{d\psi}, \quad \dot{c}^1 = \frac{dc^1}{dt}$$

$$-B_3{}^k\dot{c}^l\Gamma^3_{kl} = -B_3{}^3\dot{c}^l\Gamma^3_{3l} = -\frac{1}{R}\dot{c}^1\frac{\dot{R}}{R^2}$$

$$-B_4{}^k\dot{c}^l\Gamma^4_{kl} = -B_4{}^4\dot{c}^l\Gamma^4_{4l} = -\frac{1}{R}\dot{c}^1\frac{\dot{R}}{R^2}$$

Hence for all α, β, δ, ϵ we have

$$\dot{B}_\beta{}^\alpha = -B_\beta{}^\delta\dot{c}^\varepsilon\Gamma^\alpha_{\delta\varepsilon}$$

and so $h \circ c$ is horizontal in O^+M when c is horizontal in O^+N. ■

3.3. We are now in a position to show that the derivative of our injection, $Dh: TO^+N \to TO^+M$, preserves the Schmidt metric.

Lemma. For all $Y \in TO^+N$

$$\|Y\|_N = \|Dh(Y)\|_M$$

where the norms are derived from the Schmidt metrics for O^+N and O^+M, respectively.

Proof. We give a different proof from Bosshard who used standard horizontal and vertical fields (see I.4.2 and I.5.5). It is of course sufficient to work with the norms because of the symmetry and bilinearity of any metric tensor field, which yields the polarization identity (see [16], p. 104).

Let $x \in N$ and $(x, b_j{}^i) \in O^+N$ with an arbitrary vector $Y = (x, b_j{}^i, X^i, B_j{}^i) \in T_xO^+N$. The map Dh takes the form

$$Dh: (x, b_j{}^i, X^i, B_j{}^i) \mapsto (x, y_0, b_\beta{}^\alpha, X^\alpha, B_\beta{}^\alpha)$$

where $\alpha, \beta = 1, 2, 3, 4$ and

$$X^3 = X^4 = 0; \qquad b_3{}^3 = (R\sin\sigma)^{-1}, \qquad b_4{}^4 = (R\sin\theta\sin\sigma)^{-1}$$

$$B_3{}^3 = B_4{}^3 = B_3{}^4 = B_4{}^4 = b_4{}^3 = b_3{}^4 = 0$$

From II.1.1 for connection form ω and canonical one-form Θ the Schmidt norm is given by

$$\begin{aligned}\|Y\|_N{}^2 &= \|\Theta(Y)\|_0{}^2 + \|\omega(Y)\|_0{}^2 \\ &= \|[b_j{}^i]^{-1}[X^i]\|_0{}^2 + \|[B_j{}^i + b_j{}^k\Gamma^i_{kl}X^l][b_j{}^i]^{-1}\|_0{}^2\end{aligned}$$

by I.4.5 and I.5.6, Example 2. A similar expression holds for $\|Dh(Y)\|_M{}^2$, with all indices running through 1, 2, 3, 4.

Suppose that Y is horizontal; then $\omega(Y) = 0$ and by 3.2 we know that $Dh(Y)$ is horizontal in O^+M. Hence $\|Y\|_N{}^2 = \|Dh(Y)\|_M{}^2$ because $[X^\alpha] = [X^1, X^2, 0, 0]$.

On the other hand, if Y is vertical then $\Theta(Y) = 0$ so $[X^i] = 0$. Hence $Dh(Y)$ is vertical in O^+M and $\|Y\|_N{}^2 = \|Dh(Y)\|_M{}^2$ because

$$[B_\beta{}^\alpha] = \left[\begin{array}{c|c} B_j{}^i & \\ \hline & 0 \end{array}\right]$$

Thus our result follows because horizontal and vertical vectors are orthogonal. ■

Corollary. For all $p, q \in O^+N$

$$d_M(h(p), h(q)) \leqslant d_N(p, q)$$

where d_M and d_N are the topological metrics induced on O^+M and O^+N, respectively, by the Schmidt norm (see I.2.4). ■

We find an expression for the Riemannian metric on O^+N as follows. Consider any curve

$$c\colon [a, b] \to O^+N\colon t \mapsto (c^1(t), c^2(t), \chi_i{}^j(t))$$

with tangent vector field

$$\dot{c}\colon [a, b] \to TO^+N\colon t \mapsto (\dot{c}^1(t), \dot{c}^2(t), \dot{\chi}_i{}^j(t))$$

Here the orthonormal frame determined by $\chi_i{}^j(t)$ at $c(t)$ is $(\chi_1{}^j\partial_j, \chi_2{}^j\partial_j)$ and from 3.1 there is $\chi\colon [a, b] \to \mathbb{R}$ such that

$$[\chi_i{}^j] = \frac{1}{R \circ c^1}\begin{bmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{bmatrix}$$

For the Schmidt norm (see II.1.1) we have

$$\|\dot{c}\|_N{}^2 = \|\Theta(\dot{c})\|_0{}^2 + \|\omega(\dot{c})\|_0{}^2$$

$$\Theta(\dot{c}) = [\chi_i{}^j]^{-1}[\dot{c}^i] = R\begin{bmatrix} \dot{c}^1\cosh\chi - \dot{c}^2\sinh\chi \\ \dot{c}^2\cosh\chi - \dot{c}^1\sinh\chi \end{bmatrix}$$

$$[\dot{\chi}_i{}^j][\chi_i{}^j]^{-1} = -\frac{\dot{R}\dot{c}^1}{R}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \dot{\chi}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{where } \dot{\chi} = \frac{d\chi}{dt}, \dot{R} = \frac{dR}{d\psi}$$

$$[\chi_j{}^k\Gamma^i_{kl}\dot{c}^l][\chi_j{}^i]^{-1} = \frac{\dot{R}}{R}\begin{bmatrix} \dot{c}^1 & \dot{c}^2 \\ \dot{c}^2 & \dot{c}^1 \end{bmatrix}$$

$$\omega(\dot{c}) = [\dot{\chi}_i{}^j + \chi_j{}^k\Gamma^i_{kl}\dot{c}^l][\chi_j{}^i]^{-1} = \begin{bmatrix} 0 & \dot{\chi} + \dfrac{\dot{R}}{R}\dot{c}^2 \\ \dot{\chi} + \dfrac{\dot{R}}{R}\dot{c}^2 & 0 \end{bmatrix}$$

$$\tfrac{1}{2}\|\dot{c}\|_N{}^2 = R^2\Big(((\dot{c}^1)^2 + (\dot{c}^2)^2)\cosh 2\chi - 2\dot{c}^1\dot{c}^2\sinh 2\chi\Big) + \left(\dot{\chi} + \frac{\dot{R}}{R}c^2\right)^2$$

We can discard the factor 2, so in ψ, σ, χ coordinates the expression for the arc length formula of the Schmidt metric on O^+N is

$$ds^2 = R^2(\psi)\left((d\psi^2 + d\sigma^2)\cosh 2\chi - 2\,d\psi\,d\sigma \sinh 2\chi\right) + \left(\frac{\dot{R}(\psi)}{R(\psi)}\,d\sigma + d\chi\right)^2$$

3.4. The Initial Singularity at $\psi = 0$. We construct a nonconvergent Cauchy sequence s in (O^+N, d_N) to determine a point of the completion $\bar{O}^+N$ and hence obtain a point of the b-boundary ∂N. In consequence of 3.3, $h \circ s$ is a Cauchy sequence in O^+M, which determines a point of $\bar{O}^+M$ and so gives a point of ∂M.

Let σ_0 be fixed in S^1 and consider the sequence

$$s_1 \colon \mathbb{N} \to O^+N \colon n \mapsto (\psi_n, \sigma_0, 0)$$

for some $\psi_n \in (0, T)$ with $\psi_n \to 0$. Then for all n greater than some $N_0 \in \mathbb{N}$ we can find $t_n \in [0, 1)$ such that $\psi_n = 1 - t_n$ and $t_n \to 1$. So our sequence eventually lies on the curve

$$c \colon [0, 1) \to O^+N \colon t \mapsto (1 - t, \sigma_0, 0)$$

This curve is horizontal, so we find from 3.3

$$\|\dot{c}(t)\|_N = R(1 - t)$$

By hypothesis, there is a $\rho > 0$ such that

$$R(\psi) \sim \psi^\rho, \quad \text{for } \psi \to 0$$

Therefore, for large enough $k, n \in \mathbb{N}$,

$$d_N(s_1(n), s_1(k)) \leqslant \left|\int_{t_n}^{t_k} \|\dot{c}(t)\|_N\,dt\right| \sim \left|\int_{\psi_n}^{\psi_k} \psi^\rho\,d\psi\right|$$

Since $\psi_n \to 0$ as $n \to \infty$, for all $\epsilon > 0$ we can find $N_\varepsilon \in \mathbb{N}$ such that

$$d_N(s_1(n), s_1(k)) < \epsilon \quad \text{for } n, k > N_\varepsilon$$

Thus s_1 is a Cauchy sequence, and the equivalence class to which it belongs determines a point $\tilde{x}_0$ as limit in $\bar{O}^+N$. This projects to a point $x_0 = \Pi_{\bar{O}}(\tilde{x}_0) \in \partial N$, which we can identify as the singularity with coordinates $\psi = 0$ and $\sigma = \sigma_0$.

By a similar argument, the Cauchy sequence

$$s \colon \mathbb{N} \to O^+N \colon n \mapsto (\psi_n, \sigma_0 + \delta_n, \chi_1)$$

with $\psi_n \to 0$, $\delta_n \to 0$ and fixed σ_0, χ_1 also determines a point $\tilde{x}_1 \in \bar{O}^+N$. Moreover, $\tilde{x}_1$ and $\tilde{x}_0$ belong to the same fibre of $\bar{O}^+N$, that is, $\Pi_{\bar{O}}(\tilde{x}_1) = \Pi_{\bar{O}}(\tilde{x}_0) = x_0 \in \partial N$.

3.5. The Degeneracy of Fibres in $\bar{O}^+N$ over $\psi = 0$. To establish the degeneracy we construct for each $n \in \mathbb{N}$ a curve k_n that joins $s_1(n) = (\psi_n, \sigma_0, 0)$ and $s(n) = (\psi_n, \sigma_0 + \delta_n, \chi_1)$, for fixed σ_0 and χ_1. Then we show that the length of k_n tends to zero as $n \to \infty$.

Let $R_n = R(\psi_n)$, $\dot{R}_n = \dot{R}(\psi_n)$ and $\delta_n = -\chi_1 R_n/\dot{R}_n$. Then the curve

$$k_n: [0, \delta_n] \to O^+N: t \mapsto (\psi_n, \sigma_0 + t, -(\dot{R}_n/R_n)t)$$

is horizontal by 3.3. It joins $s(n)$ and $s_1(n)$ as required, and its length is

$$K_n = \left| R_n \int_0^{\delta_n} \left(\cosh\left(-2\frac{\dot{R}_n}{R_n}t\right)\right)^{1/2} dt \right| = \left| \frac{R_n^{\,2}}{\dot{R}_n} \int_0^{\chi_1} (\cosh 2\chi)^{1/2}\, d\chi \right|$$

We can obtain close bounds on this via the inequalities

$$0 < \cosh\chi \leqslant (\cosh^2\chi + \sinh^2\chi)^{1/2} = (\cosh 2\chi)^{1/2} \leqslant 2^{1/2}\cosh\chi$$

Hence, and because $\psi_n \to 0$ as $n \to \infty$,

$$K_n \leqslant |2^{1/2}(R_n^{\,2}/\dot{R}_n)\sinh\chi_1| \to |2^{1/2}\sinh\chi_1\psi_n^{\rho+1}|$$

which tends to zero as $n \to \infty$ because by hypothesis we have $\rho > 0$.

Therefore $d_N(s(n), s_1(n))$ tends to zero as $n \to \infty$ so in the limit $\bar{d}_N(\tilde{x}_0, \tilde{x}_1) = 0$ and $\tilde{x}_0 = \tilde{x}_1$. But σ_0 and χ_1 were arbitrary, so we may conclude that any fibre over the singularity at $\psi = 0$ is degenerate.

3.6. The subspace of $\bar{O}^+N$ over $\psi = 0$ is a point and $\bar{N}$ is non-Hausdorff.

Proof (Johnson [38], independently of Bosshard's work). We define the *essential boundary* of N as $\Pi_{\bar{O}}(\hat{O}^+N)$ where

$$\hat{O}^+N = \left\{ \lim_{t\to t_2} c(t) \in \bar{O}^+N \,|\, c: [t_1, t_2) \to O^+n \text{ is inextensible, finite, with } c^1(t) \text{ not bounded away from zero} \right\}$$

We also write $c_0: (\psi, \sigma_0, 0)$ for the "radial" curves with fixed $\sigma_0 \in S^1$; these are horizontal lifts of their projections. Let

$$\tilde{x}_0(\sigma_0) = \lim_{\psi\to 0} c_0(\psi)$$

$$P_0 = \{\tilde{x}_0(\sigma_0) | \sigma \in S^1\}$$

We outline the proof given in detail by Johnson [38].

Lemma 1. P_0 is compact, because the map $S^1 \to P_0: \sigma_0 \mapsto \tilde{x}_0(\sigma_0)$ is continuous.

Lemma 2. Each fibre $\Pi_{\bar{O}}^{\leftarrow} \circ \Pi_{\bar{O}}(\tilde{x}_0(\sigma_0))$ reduces to a point. (This is our result 3.5 from Bosshard [7].)

Lemma 3. For all $\epsilon > 0$ and all $\sigma_0 \in S^1$

$$\bar{d}_N(\tilde{x}_0(0), \tilde{x}_0(\sigma_0)) < \epsilon$$

(This follows from our proof of 3.5 by a suitable adjustment of the sequence of curves k_n.)

Hence we may conclude that P_0 consists of one point, $\tilde{x}_0$.

Lemma 4. $\hat{O}^+N = \{\tilde{x}_0\}$, so the essential boundary of N is a point.

We can see this from our previous study of horizontal curves. Let

$$\tilde{c}: [0, \bar{s}) \to O^+N: s \mapsto (\tilde{c}^1(s), \tilde{c}^2(s), \tilde{\chi}(s))$$

be any curve in O^+N such that there is a sequence $s_n \to \bar{s}$ with $\psi_n = \tilde{c}^1(s_n) \to 0$; it matters not whether $\chi_n = \tilde{\chi}(s_n) \to \infty$.

Now consider the horizontal curves

$$c_n: (0, \psi_n] \to O^+N: t \mapsto (t, \sigma_n + t - \psi_n, \chi_n - \log R/R_n)$$

defined for all $n \in \mathbb{N}$ and such that $c_n(\psi_n) = \tilde{c}(s_n)$, so $R_n = R(\psi_n)$, $\sigma_n = \tilde{c}^2(s_n)$. By 3.3 the length of c_n is

$$C_n = \int_0^{\psi_n} \|\dot{c}_n(t)\|_N \, dt = \left| 2^{1/2} \frac{e^{-\chi_n}}{R_n} \int_0^{\psi_n} R^2(t) \, dt \right|$$

Hence, as $\psi_n \to 0$ and $R(\psi_n) \sim \psi_n{}^\rho$ for some $\rho > 0$,

$$C_n \sim 2^{1/2} e^{-\chi_n} \psi_n^{\rho+1} \to 0$$

But, for any fixed n,

$$\lim_{t \to 0} c_n(t) = \tilde{x}_0$$

because C_n is bounded, the radial distance from $c_n(t)$ to $\tilde{x}_0(\sigma_n + t - \psi_n)$ tends to zero as $t \to 0$, and $P_0 = \{\tilde{x}_0\}$ is compact

Lemma 5. The only open set in $\bar{N}$ that contains $\Pi_{\bar{o}}(\tilde{x}_0)$ is $\bar{N}$ itself, so $\Pi_{\bar{o}}(\tilde{x}_0)$ cannot be Hausdorff separated from any point of N.

This follows from the fact that for arbitrary $(\psi_0, \sigma_0) \in N$ the vertical sequence

$$s: \mathbb{N} \to O^+N: n \mapsto (\psi_0, \sigma_0, n)$$

converges to $\tilde{x}_0$. For, consider the curves

$$f_n: (0, \psi_0] \to O^+N: t \mapsto (t, \sigma_0 + t - \psi_0, n - \log R/R_0)$$

with $R_0 = R(\psi_0)$. As in Lemma 4 these have length

$$F_n = \left| 2^{1/2} \frac{e^{-n}}{R_0} \int_0^{\psi_0} R^2(t) \, dt \right|$$

which tends to zero as $n \to \infty$. Therefore $\lim s(n) = \tilde{x}_0$ and so $\Pi_0 \circ s(n) = (\psi_0, \sigma_0) \to \Pi_{\bar{o}}(\tilde{x}_0) \in \bar{N}$. ■

This completes the proof of the stated proposition, but in fact Johnson proved considerably more, namely, the following generalization.

Generalization 1. The preceding result holds for any two-manifold

$$(0, r_m) \times S^1, \quad \text{for any } r_m < \infty$$

with metric given by

$$ds^2 = -b^2(r)\,dr^2 + a^2(r)\,d\sigma^2$$

where a and b are positive smooth functions $(0, r_m) \to \mathbb{R}^+$ satisfying the conditions:

(i) $\dot{b} \geqslant 0$ on $(0, r_m)$;
(ii) $\lim_{r\to 0^+} b(r)/\dot{a}(r) = 0$ and $d(b/\dot{a})/dr \geqslant 0$ on $(0, r_m)$;
(iii) $a(r)/\dot{a}(r) \geqslant rC$ on $(0, r_m)$ for some constant $C > 0$;
(iv) for all $r_0 \in (0, r_m)$ there are smooth positive functions $\tilde{a}, \tilde{b}$: $\mathbb{R} \to \mathbb{R}^+$ such that

$$\tilde{a}|_{[r_0,r_m]} = a, \qquad \tilde{b}|_{[r_0,r_m]} = b$$

and $\tilde{a}(r) = 1 = \tilde{b}(r)$ except on a compact set;
(v) for some sequence $x_n \to 0$ on $(0, r_m)$

$$\sum_{n=1}^{\infty} \int_0^{x_n} b(r)\,dr < \infty, \qquad \sum_{n=1}^{\infty} \int_0^{x_n} \frac{b(r)}{a(r)}\,dr = \infty$$

These conditions are satisfied by the Friedmann two-manifold N that we have been studying. They are also satisfied by the corresponding two-manifold from Schwarzschild space–time, for which

$$a(r) = r, \qquad b(r) = (2/r - 1)^{-1/2} \quad \text{with } r_m = 1$$

In all cases, when conditions (1)–(v) are met, there is a singularity at "$r = 0$," and the b-boundary is essentially a point whose only neighborhood is the whole completed manifold.

Generalization 2. The bundle-completions of four-dimensional Friedmann and Schwarzschild space–times are non-Hausdorff.

This follows from the existence of an injective isometry (like h in 3.3) of the appropriate two-manifold into the full space–time in each case. Horizontal lifts of radial ($r \to 0$) curves lying in the image of the injection again yield a single point, like P_0 in Lemma 3. Every neighborhood of the projection of this point by $\Pi_{\bar{O}^+M}$ into the full space–time contains the image of the injection, because, as in Lemma 5, arbitrary constant sequences in that image also converge to the boundary point. The isometric action of the orthogonal group allows us to rotate the image of our injected two-manifold. So, for all points y of the space–time there is a b-boundary point x such that y is in every neighborhood of x.

3.7. Closed Friedmann Space–Time. Plainly, if for some $T \in \mathbb{R}$ we have also $R(\psi) \to 0$ as $\psi \to T$, and $R(T - x) \sim x^{\eta}$ for some $\eta > 0$ as $x \to 0$, then there is another singularity corresponding to $\psi = T$. It will be entirely similar to the initial singularity at $\psi = 0$. Cosmologically, such a situation corresponds to an initial big bang followed by universal expansion, contraction, then a final complete collapse. Such a model is called a *closed* Friedmann space–time.

3.7.1. Sequences s_1, s_2 for Initial and Final Singularities. For definiteness we can think of the particular model (see I.5.10) given by

$$R\colon (0, 2\pi) \to \mathbb{R}\colon \psi \mapsto 1 - \cos\psi$$

In this case $T = 2\pi$, but for our purposes we need only the property that for $x \to 0$

$$R(x) \to R(2\pi - x) \sim x^{\rho}, \quad \text{for some } \rho > 0$$

We have a Cauchy sequence

$$s_1\colon \mathbb{N} \to O^+N\colon n \mapsto (\psi_n, \sigma_0, 0) \quad \text{with } \psi_n \to 0$$

and with limit $\tilde{x}_0 \in \bar{O}^+N$. Likewise we can find a similar Cauchy sequence

$$s_2\colon \mathbb{N} \to O^+N\colon n \mapsto (2\pi - \psi_n, \sigma_0 + 2\pi - 2\psi_n, 0)$$

with limit $\tilde{x}_2 \in O^+N$. Then $x_0 = \Pi_{\bar{O}}(\tilde{x}_0)$ and $x_2 = \Pi_{\bar{O}}(\tilde{x}_2)$ are b-boundary points corresponding to the initial and final singularities, respectively. Our plan now is to find a piecewise-smooth curve that joins $s_1(n)$ to $s_2(n)$ for each $n \in \mathbb{N}$ and has zero length in the limit $n \to \infty$. By that means we obtain $d_N(\tilde{x}_0, \tilde{x}_2) = 0$, so $\tilde{x}_0 = \tilde{x}_2$ and then $x_0 = x_2$; hence the initial and final singularities are identified in the b-boundary. (See the diagram in 3.7.4.)

3.7.2. Curves l_n, j_n up the Fibres through s_1, s_2. Consider the sequence

$$r_1\colon \mathbb{N} \to O^+N\colon n \mapsto (\psi_n, \sigma_0 - (R_n/\dot{R}_n)\chi_n, \chi_n)$$

where $\psi_n \to 0$, and $\chi_n = -\delta \log R_n$ for some $\delta > 1$ to be fixed later. A curve l_n, which joins $s_1(n)$ and $r_1(n)$ for each n, is given by

$$l_n\colon [0, \alpha_n] \to O^+N\colon t \mapsto (\psi_n, \sigma_0 + t, -(\dot{R}_n/R_n)t)$$

so

$$\alpha_n = -\frac{R_n}{\dot{R}_n}\chi_n = \delta \frac{R_n}{\dot{R}_n} \log R_n$$

Hence for $\psi_n \to 0$ as $n \to \infty$ we have

$$\chi_n \to \infty \quad \text{and} \quad \alpha_n \sim \psi_n \log \psi_n \to 0$$

Thus, $\lim_{n\to\infty} r_1(n)$ lies in the same fibre of $\bar{O}^+N$ as $\tilde{x}_0 = \lim_{n\to\infty} s_1(n)$.

From 3.5 we know that l_n is horizontal with length

$$L_n \leqslant \left| 2^{1/2} \frac{R_n{}^2}{\dot{R}_n} \sinh \chi_n \right| = 2^{-1/2} |(R_n^{2-\delta} - R_n^{2+\delta})/\dot{R}_n|$$

But, $R_n \sim \psi_n{}^\rho$ so $\dot{R}_n \sim \psi_n^{\rho-1}$ for $n \to \infty$, so

$$L_n \sim |\psi_n^{1+\rho-\delta\rho} - \psi_n^{1+\rho+\delta\rho}|$$

which tends to zero for any δ such that $1 < \delta < (1 + \rho)/\rho$.

We can find a precisely similar curve j_n for each n that joins $s_2(n)$ to $r_2(n)$ where

$$r_2 \colon \mathbb{N} \to O^+N \colon n \mapsto (2\pi - \psi_n, \sigma_0 - \chi_n(R_n/\dot{R}_n) + 2\pi - 2\psi_n, \chi_n)$$

Again the length J_n of j_n tends to zero as $n \to \infty$, so the fibres over $\psi = 2\pi$ are degenerate, as we observed before.

3.7.3. A curve Joining Initial and Final Fibres. Next we take a curve, for each n, that joins $r_1(n)$ to $w(n)$ where

$$w \colon \mathbb{N} \to O^+N \colon n \mapsto \left(2\pi - \psi_n, \sigma_0 - \chi_n \frac{R_n}{\dot{R}_n} + 2\pi - 2\psi_n, \chi_n + \log \frac{R_n}{R(2\pi - \psi_n)}\right)$$

Such a curve is the horizontal lift of a null geodesic given by

$$\begin{aligned} m_n \colon & [\psi_n, 2\pi - \psi_n] \to O^+N \\ \colon & t \mapsto \left(t, \sigma_0 - \chi_n \frac{R_n}{\dot{R}_n} - \psi_n + t, \chi_n + \log \frac{R_n}{R(t)}\right) \end{aligned}$$

The length of m_n is

$$M_n = \left| \int_{\psi_n}^{2\pi-\psi_n} 2^{1/2} R(t) e^{-\chi(t)}\, dt \right| = 2^{1/2} \left| R_n^{\delta-1} \int_{\psi_n}^{2\pi-\psi_n} R^2(t)\, dt \right|$$

We are assured by the properties of R that there exists some $A \in \mathbb{R}$ with

$$\lim_{x \to 0} \int_x^{2\pi - x} R^2(t)\, dt = A$$

Hence

$$M_n < 2^{1/2} A R_n^{\delta-1} \sim 2^{1/2} A \psi_n^{\rho(\delta-1)}$$

which tends to zero as $\psi_n \to 0$ since $\rho > 0$ and $1 < \delta < (1 + \rho)/\rho$.

The final stage is to provide a curve for each n that joins $r_2(n)$ to $w(n)$. This is achieved by the vertical curve

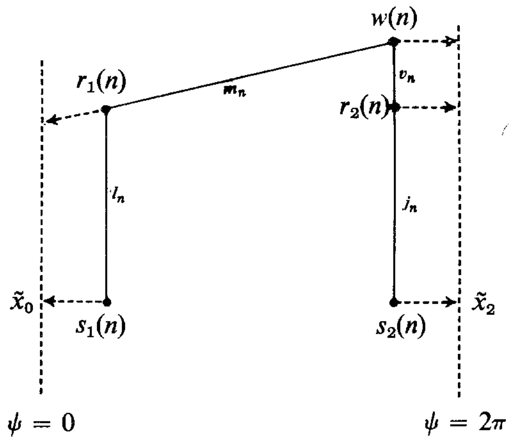
$$\begin{aligned} v_n \colon & [0, \log (R_n/R(2\pi - \psi_n))] \to O^+N \\ \colon & t \mapsto (2\pi - \psi_n, \sigma_0 - \chi_n(R_n/\dot{R}_n) + 2\pi - 2\psi_n, \chi_n + t) \end{aligned}$$

From the arc length formula for γ in 3.3, v_n has length

$$v_n = |\log(R_n/R(2\pi - \psi_n))|$$

which tends to zero as $\psi_n \to 0$ because we have supposed that then $R_n \to R(2\pi - \psi_n)$.

3.7.4. Implications. Our various curves and sequences are summarized in the following diagram.



For all n we have

$$d_N(s_1(n), s_2(n)) \leqslant L_n + J_n + M_n + V_n = F_n$$

and so by Corollary 3.3

$$d_M(h \circ s_1(n), h \circ s_2(n)) \leqslant F_n$$

But $F_n \to 0$ as $n \to \infty$. Also s_1 and s_2 are Cauchy sequences on O^+N. Hence $h \circ s_1$ and $h \circ s_2$ are Cauchy sequences on O^+M, belonging to the same equivalence class in the completion space $\bar{O}^+M$. Therefore points of the initial and final singularities are identified in the bundle completion $\overline{M} = M \cup \partial M$.

This result casts doubt on the physical significance of the b-boundary, for the two singularities arise in strikingly different physical regimes: expansion and contraction. Since we have put no causal structure into the completion process, we can hardly expect the geometry to take account of the direction of evolution, but we would hope for temporally, spatially, and physically remote occurrences to be kept apart. There is of course a further problem of like kind, arising from the work of Johnson that we discussed in

3.6. The non-Hausdorff nature of the bundle completion leads to the impression that events remote from a singularity are in every neighborhood of it, that is topologically close to it.

The crucial stage in Bosshard's identification of initial and final singular points is the establishment of degeneracy in the fibres over them. That makes it possible to lift an apparently innocuous null geodesic arbitrarily high up O^+N and so join the degenerate fibres with arbitrarily small bundle length. Consider at any point in N a choice of orthonormal frame represented by $\chi = \chi_0$. Physically [16] it corresponds to a choice of observer (or reference system) with velocity v_0 along the ψ-axis (relative to the observer using the frame $\chi = 0$, representing $(1/R)[\partial_\psi, \partial_\sigma]$, at the point) where

$$\cosh \chi_0 = (1 - v_0{}^2)^{-1/2}$$

So $v_0 \to 0$ as $\chi_0 \to 0$; and $v_0 \to 1$ as $x_0 \to \infty$, which means that this observer approaches the speed of light relative to the former observer. The essential geometry is captured in the submanifold

$$N_0 = \{(v, v) \in N\} \cong (0, 2\pi)$$

It has a bundle of null frames with structure group $\mathbb{R}^+$. Let L^+N_0 be the connected component of the identity, and at all $\nu \in (0, 2\pi)$ let the frame corresponding to $1 \in \mathbb{R}^+$ be

$$\tilde{\partial}_1 = \tfrac{1}{2}(\partial_\psi + \partial_\sigma)$$

The connection $\tilde{\nabla}$ induced by the Levi–Cività connection ∇ on N is given in components by $\tilde{\Gamma}^1_{11}$ where

$$\tilde{\nabla}_{\tilde{\partial}_1}\tilde{\partial}_1 = \tilde{\Gamma}^1_{11}\tilde{\partial}_1 = \tfrac{1}{4}\nabla_{\partial_\psi + \partial_\sigma}(\partial_\psi + \partial_\sigma) = \frac{\dot{R}}{R}\tilde{\partial}_1$$

Then a typical curve

$$c_n\colon (0, 2\pi) \to N_0\colon t \mapsto t$$

has a horizontal lift through any $\lambda_n \in \mathbb{R}^+$ at $t = \pi$, say, given by

$$c_n{}^\uparrow\colon (0, 2\pi) \to L^+N_0\colon t \mapsto (t, \lambda(t))$$

where the positive function λ satisfies

$$\frac{1}{\lambda}\frac{d\lambda}{dt} = -\frac{\dot{R}}{R}, \qquad \lambda(\pi) = \lambda_n$$

So, for $t \in (0, 2\pi)$, $\lambda(t) = \lambda_n R(\pi)/R(t)$.

The Schmidt norm gives

$$\|\dot{c}_n{}^\uparrow\|_{N_0} = \frac{1}{\lambda_n}\frac{R(t)}{R(\pi)}$$

From the properties of R we are assured of a finite limit

$$\lim_{x\to 0}\int_x^{2\pi - x} R(t)/R(\pi)\,dt = B$$

Therefore the length of $c_n{}^\dagger$ is B/λ_n, which we can make tend to zero by choosing $\lambda_n \to \infty$ as $n \to \infty$. This kind of behavior is likely to be quite common for space–times: choosing frames having increasingly large components with respect to a given standard basis makes the components of a given vector become increasingly small. The property of possessing, in a continuous fashion, a standard basis at each point is called (I.2.7) *parallelizability*. Though rare among manifolds in general, it is quite common for simple space–times. We shall take advantage of this property to modify the Schmidt metric for the Friedmann two-manifold N, so that the degeneracy in fibres over the singular points can be removed. This leads to a separation of the initial and final singularities. The particular case for Friedmann space–time is discussed in 3.9. First, we give a general definition and prove some properties for a parallelizable manifold. We assume our parallelization to be continuously differentiable. Simply connected space–times always admit a parallelization if they are space- and time-oriented (see Lee [46] and Geroch [27]).

3.8. A New Bundle Metric for Parallelizable Manifolds. Let M be a smooth n-manifold possessing a continuously differentiable parallelization

$$p\colon M \to L'M\colon x \mapsto (p_i)_x$$

where $L'M$ is a connected component of LM with structure group G^+.

Proposition 1. Every chart about any $x \in M$ induces a chart about $p(x) \in L'M$.

Proof. By taking if necessary a subchart (U, φ) about $x \in M$, we have by the bundle structure (see I.4.1) a diffeomorphism

$$\Pi_{L'}^{\leftarrow}(U) = L'U \cong U \times G^+$$

We denote by $(p_i)_y$ the frame determined by $p(y)$ at $y \in U$. Then for any $g \in G^+$ we can find a matrix representation $[g_j{}^i]_{p(y)}$ for the basis it determines at y, by reference to the $(p_i)_y$. This gives the desired chart $(L'U, \Phi_p)$ by

$$\Phi_p\colon L'U \cong U \times G^+ \to \mathbb{R}^n \times \mathbb{R}^{n^2}$$
$$\colon (y, (g_j{}^i p_i)) \cong (y, g) \mapsto (\varphi(y), [g_j{}^i]_{p(y)}) \quad \blacksquare$$

We have done the obvious thing: chosen $p(y)$ as our reference frame for T_yM, in the fibre $\Pi_{L'}^{\leftarrow}(y)$. This implies that $\Phi_p(y, p(y)) = (\varphi(y), I)$, where I is the unit matrix.

Plainly, Φ_p has the same class as p on U.

Definition 1. Denote by $(\partial_i)_y$ the basis for T_yM induced for all $y \in U$ by the chart (U, φ) and denote by $(\Delta_j{}^i)_g$ the basis induced for T_gG^+ by the coordinatization

$$G^+ \to \mathbb{R}^{n^2}: g \mapsto [g_j{}^i]_{p(y)}$$

Then the basis $((\partial_i)_y, (\Delta_j{}^i)_g)$ is induced for $T_{(y,g)}L'M$ by the chart $(L'U, \Phi_p)$.

The *vertical form* for $L'M$ with selection p is the map

$$\begin{aligned}\tilde{p}&: TL'M \to \mathbb{R}^{n^2}\\ &: (\varphi(y), [g_j{}^i]_{p(y)}, A^i\partial_i, B_i{}^j\Delta_j{}^i) \mapsto [B_i{}^j][g_j{}^i]^{-1}_{p(y)}\end{aligned}$$

Remark. This map is independent of the choice of chart (U, φ) and it is linear on each tangent space $T_{(y,g)}L'M$. Also if p is of class C^k for some $k > 0$, then $\tilde{p}$ is of class $k - 1$. The algebraic dependence of $\tilde{p}$ on p is considered next.

> *Proposition 2.* Suppose that $q: M \to L'M$ is also a continuously differentiable parallelization. Then there exists a continuously differentiable map
>
> $$[a_j{}^i]: M \to \mathbb{R}^{n^2}: y \mapsto [a_j{}^i]_y$$
>
> such that
>
> $$\Phi_q \circ \Phi_p^{-1} = (I, [a_j{}^i])$$

Proof. By transitivity of the action of G^+ on $L'M$ we can find a C^1 map $a: M \to G^+$ such that for all $x \in M$

$$q(x) = R_{a(x)}(p(x))$$

Hence the required matrix field is given by $p_k = a_k{}^i p_i$. ■

> *Corollary* (for which the author is indebted to C. J. S. Clarke).
>
> $$\tilde{q}(X) = \tilde{p}(X) + [A^l(\partial_l a_k{}^i)][a_k{}^i]^{-1}$$
>
> for all
>
> $$X = (\varphi(y), [g_j{}^i]_{p(y)}, A^i\partial_i, B_i{}^j\Delta_j{}^i) \in TL'M$$

Proof. The derivative of $\Phi_q \circ \Phi_p^{-1}$ is the Jacobian

$$\begin{bmatrix} \delta_j{}^i & 0 \\ \partial_l a_j{}^i & a_j{}^i \end{bmatrix}, \quad \text{with } I = [\delta_j{}^i] \quad ■$$

Remark. Consider a curve $c: (0, 1) \to L'M$ which appears in coordinates via $(L'U, \Phi_p)$ as

$$\Phi_p \circ c: t \mapsto (c^i(t), [g_j{}^i]_t), \quad \text{for } c(t) \in L'U$$

with tangent vector field

$$\dot{c}: t \mapsto (c^i(t), [g_j{}^i]_t, \dot{c}^i(t), [\dot{g}_j{}^i]_t)$$

The vertical form gives a logarithmic measure of vertical velocity:

$$\tilde{p}(\dot{c}) = [\dot{g}_j{}^i][g_j{}^i]^{-1}$$

It is precisely this feature that prompts the following modification of the Schmidt metric given in II.1.1.

Definition 2. The modified Riemannian metric on $L'M$ is (see II.1.1)

$$\langle\ ,\ \rangle_p : TL'M \times TL'M \to \mathbb{R} : (X, Y) \mapsto \langle X, Y\rangle + \tilde{p}(X)\cdot\tilde{p}(Y)$$

where $\langle X, Y\rangle = \Theta(X)\cdot\Theta(Y) + \omega(X)\cdot\omega(Y)$ is the Schmidt metric and $\cdot$ is the standard inner product on $\mathbb{R}^n$ and $\mathbb{R}^{n^2}$. Clearly $\langle\ ,\ \rangle_p$ is well defined and a Riemannian metric tensor field on $L'M$. Let d_p be the topological metric induced on $L'M$ by $\langle\ ,\ \rangle_p$ via its norm $\|\ \|_p$. Evidently, $\langle\ ,\ \rangle_p$ has the same class of differentiability as $\tilde{p}$, which is governed by the class of parallelization.

Proposition 3. (i) If we replace the standard inner product $\cdot$ on $\mathbb{R}^n$ and $\mathbb{R}^{n^2}$ by any other inner products, then the ensuing metric structure is uniformly equivalent to that given by d_p.

(ii) For all $g \in G^+$ the right action $R_g : L'M \to L'M$ is uniformly continuous on $(L'M, d_p)$. Hence R_g has a unique uniformly continuous extension to the Cauchy completion $(\tilde{L}'M, \tilde{d}_p)$ in which $(L'M, d_p)$ is dense.

(iii) The topological space $\tilde{L}'M/G^+ = \tilde{M}$ is well defined, and we shall call $\tilde{\partial}M = \tilde{M}\backslash M$ the *p-boundary* of M.

(iv) If the connection on LM is the Levi–Cività connection of some metric g, then we can replace G^+ by $\hat{O}^+$ and work with O^+M and obtain $\tilde{\partial}M = (\tilde{O}^+M/\hat{O}^+)\backslash M$.

Proof. See [15] for details.

(i) This follows from II.1.2.

(ii) This follows from II.1.3 and the observation that for all $X \in TL'M$ and all $g \in G^+$

$$\|\tilde{p} \circ DR_g(X)\|_0 = \|\tilde{p}(X)\|_0$$

(iii) This follows from I.3.9.

(iv) This follows because $\tilde{p}$ is essentially unaltered by passing to a subbundle. ■

We defer a detailed study of the p-boundary, merely noting immediate properties in the following two propositions and thereafter describing the new criterion for completeness in Definition 3. Geroch [27] proved that a space–time admits a parallelization p if and only if it admits a spinor structure for which K. K. Lee [46] has shown that embeddability in $\mathbb{R}^6$, or simple connectedness with space- and time-orientability is a sufficient condition. That may throw some light on the naturality of our new construction. Beem [3] pointed out that b-completeness remains a stronger condition than geodesic completeness, even for the restricted class of globally hyperbolic space–times. It will have to be seen how p-completeness compares.

We shall denote by $\Pi_{\tilde{L}'}$ the projection induced by the factorization of $\tilde{L}'M$ by G^+. The topology on $\tilde{M} = \tilde{L}'M/G^+$ is such that $\Pi_{\tilde{L}'}$ is continuous. We find the following.

Proposition 4.

(i) G^+ acts transitively on fibres of $\tilde{L}'M$.

(ii) $\Pi_{\tilde{L}'}$ is an open map.

(iii) The fibres of $\tilde{L}'M$ are complete with the induced metric.

(iv) The fibres of $\tilde{L}'M$ are homogeneous spaces.

Proof. These results follow immediately from the proofs given in II.2.1–II.2.4 and Proposition 2 in Section 3.7. ■

Proposition 5. Holonomy bundles generate the same p-boundary for space–times.

The results corresponding to III.2.7 and III.2.8 hold for our new structure:

(i) If $x \in \tilde{\partial} M$ is determined by a Cauchy sequence (v_n) in $L'M$, then this x is equivalently determined by a Cauchy sequence (u_n) on a horizontal curve in $L'M$.

(ii) The holonomy bundle through any point in the frame bundle determines the same p-boundary as the frame bundle itself.

Proof. (ii) follows from (i), and III.2.7 establishes (i). ■

Definition 3. We can now modify our criterion for completeness of M via the following (see II.3.9).

(i) A curve c in M is said to have *finite p-length* if it has a horizontal lift $c^{\uparrow}$ of finite length in $(L'M, \langle\ ,\ \rangle_p)$. As before, this is independent of the choice of point in $\Pi_{\tilde{L}}^{\leftarrow}(c)$ through which the lift is effected.

(ii) A curve $c\colon [0, 1) \to M$ is called *p-incomplete* if it has finite p-length and admits no continuous extension in M to domain $[0, 1]$. Again, the definition extends trivially to any piecewise-C^1 reparameterization of the curve. Therefore, the p-boundary consists precisely of the endpoints in $\tilde{M}$ of p-incomplete curves in M.

(iii) M is called *p-complete* if it contains no p-incomplete curves, that is if $\tilde{\partial} M = \emptyset$.

3.9. Eliminating Degeneracy of Friedmann Fibres. We have shown that a Friedmann space–time is parallelizable by the existence (see 3.1) of the smooth section

$$p_M\colon M \to O^+M\colon (\psi, \sigma, \theta, \varphi) \mapsto \frac{1}{R(\psi)}\,[\partial_\psi, \partial_\sigma, (\sin\sigma)^{-1}\partial_\theta, (\sin\theta\sin\sigma)^{-1}\partial_\phi]$$

where the obvious coordinate singularities can easily be avoided. Hence $\tilde{p}_M$ and $\|\ \|_{p_M}$ are well defined on O^+M by 3.8.

Similarly, the first two terms of p_M give a smooth section p_N of O^+N, and here we find (see 3.1, 3.7)

$$\tilde{p}_N\colon TO^+N \to \mathbb{R}^{n^2}\colon \left(\psi, \sigma, L(\chi)\frac{1}{R}\begin{bmatrix}\partial_\psi\\ \partial_\sigma\end{bmatrix}, \frac{1}{R}X^i\partial_i, \frac{1}{R}B_j{}^i\partial_i\right) \mapsto [B_j{}^i][L(\chi)]^{-1}$$

where

$$O^+N = \left(\left(\psi, \sigma, L(\chi)\frac{1}{R}\begin{bmatrix}\partial_\psi\\ \partial_\sigma\end{bmatrix}\right) | (\psi, \sigma) \in N, \chi \in \mathbb{R}\right) \simeq N \times \hat{O}^+$$

and

$$L(\chi) = \begin{bmatrix}\cosh\chi & \sinh\chi\\ \sinh\chi & \cosh\chi\end{bmatrix}$$

We recall from 3.1 the smooth injection $h\colon O^+N \to O^+M$ and observe that $h \circ p_N = p_M$; also we have the counterpart of Lemma 3.3.

Lemma 1. $\|Y\|_{p_N} = \|Dh(Y)\|_{p_M}$ for all $Y \in TO^+N$.

Proof. The map Dh takes the form (see 3.3)

$$Dh\colon (x, [b_j{}^i], [X^i], [B_j{}^i]) \mapsto \left(x, y_0, [b_\beta{}^\alpha], \begin{bmatrix}X^i\\ 0\end{bmatrix}, \begin{bmatrix}B_j{}^i & 0\\ 0 & 0\end{bmatrix}\right)$$

Therefore for $Y \in TO^+N$, for some $\chi \in \mathbb{R}$,

$$\tilde{p}_N(Y) = [B_j{}^i][L(\chi)]^{-1}$$

$$\tilde{p}_M(Dh(Y)) = \begin{bmatrix}B_j{}^i & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}L(\chi) & 0\\ 0 & I\end{bmatrix}^{-1} = \tilde{p}_N(Y)$$

But from Lemma 3.3, $\|Y\|_N = \|Dh(Y)\|_M$, so the result follows. ∎

Remark. We continue to follow 3.3 and obtain an expression for the arc length formula of the new metric $\langle\ ,\ \rangle_{p_N}$ for O^+N. For any curve

$$c\colon [a, b] \to O^+N\colon t \mapsto (c^1(t), c^2(t), L(\chi(t)))$$

we have

$$\tilde{p}_N(\dot{c}) = [\dot{L}(\chi)][L(\chi)]^{-1}$$

$$= \dot{\chi}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad \text{where } \dot{\chi} = \frac{d\chi}{dt}$$

This matrix is of course independent of R, and we note that it was one of the three matrices in 3.3 that summed to give $\omega(\dot{c})$. Hence we have $\|\dot{c}\|_{p_N}$, and the arc length formula is simply

$$ds_p{}^2 = ds^2 + d\chi^2$$

where ds^2 is the expression in 3.3 for the arc length from $\langle\ ,\ \rangle$.

Lemma 2. We use p to denote p_N when no confusion arises.

(i) $s_1: \mathbb{N} \to O^+N: n \mapsto (\psi_n, \sigma_0, L(0))$ is Cauchy on (O^+N, d_p) if $\psi_n \to 0$, for any fixed $\sigma_0 \in S^1$. The limit of s_1 in $\tilde{O}^+N$ lies in the fibre over $\psi = 0$, $\sigma = \sigma_0$ in $\tilde{\partial}N$.

(ii) The fibres over $\psi = 0$ are not degenerate.

Proof. (i) This follows from the fact that s_1 eventually lies on the curve c in 3.4, and since $\dot{\chi} = 0$ on c

$$d_p(s_1(n), s_1(k)) \to d_N(s_1(n), s_1(k)) \to 0 \quad \text{as } n, k \to \infty$$

(ii) Suppose that the fibre over $\psi = 0$, $\sigma = \sigma_0$ is degenerate. By (i), the sequence

$$s^1: \mathbb{N} \to O^+N: n \mapsto (\psi_n, \sigma_0, L(\chi_0)), \qquad \sigma_0, \chi_0 \text{ fixed}$$

is Cauchy if $\psi_n \to 0$. Moreover, both $\lim s^1$ and $\lim s_1$ lie in the fibre of $\tilde{O}^+N$ over $\psi = 0$, $\sigma = \sigma_0$. By hypothesis $\tilde{d}_p(\lim s_1, \lim s^1) = 0$; so

$$\lim s_1 = \lim s^1$$

Hence we can find a sequence of curves $c_n: [0, 1] \to O^+N$ such that

$$c_n(0) = s_1(n), \qquad c_n(1) = s^1(n), \qquad \int_0^1 \|\dot{c}_n(t)\|_p \, dt \to 0$$

Therefore we must have, as $n \to \infty$,

$$\Theta(\dot{c}_n) \to 0, \qquad \omega(\dot{c}_n) \to 0, \qquad \tilde{p}(\dot{c}_n) \to 0$$

We were able to satisfy the first two of these conditions in 3.5. But from the preceding remark, $\tilde{p}(\dot{c}_n) \to 0$ if and only if $\dot{\chi}_n \to 0$. And that is possible for the functions $\dot{\chi}_n$ only if $(\chi_n(t) - \chi_n(t')) \to 0$ for all $t, t' \in [0, 1]$. Hence we require $\chi_0 = 0$. But $\lim s^1$ exists for any $\chi_0 \in \mathbb{R}$, so the fibre over $\psi = 0$, $\sigma = \sigma_0$ is not degenerate. ∎

Corollary. In the closed Friedmann submanifold N the fibres over the final singularity are not degenerate. In consequence the initial and final singularities do not have common points in the p-boundary.

Proof. The first part is clear enough, from our definition of closed Friedmann space–time.

For the second part we consider the norm of the tangent vector to an arbitrary curve c in O^+N:

$$\|\dot{c}\|_p^2 = R^2\left(((\dot{c}^1)^2 + (\dot{c}^2)^2)\cosh 2\chi - 2\dot{c}^1\dot{c}^2 \sinh 2\chi\right) + \left(\frac{\dot{R}}{R}\dot{c}^2 + \dot{\chi}\right)^2 + (\dot{\chi})^2$$

Suppose that the initial and final singularities have a common point in $\tilde{\partial}N$. Then there is a sequence of curves

$$c_n\colon [0, 1] \to O^+N$$

and $x_0, x_1 \in \tilde{\partial}N$ such that as $n \to \infty$

(i) $$\Pi_{\tilde{O}^+} c_n(0) \to x_0$$

(ii) $$\Pi_{\tilde{O}^+} c_n(1) \to x_1$$

(iii) $$\int_0^1 \|\dot{c}_n(t)\|_p \, dt \to 0$$

Now, (i) and (ii) imply that

$$\lim c_n^1(0) = 0 \neq 2\pi = \lim c_n^1(1)$$

since we suppose that $\lim c_n(0)$ is in a fibre over $\psi = 0$ and $\lim c_n(1)$ is in a fibre over $\psi = 2\pi$. Hence we cannot have $\dot{c}_n^1$ tending to the zero function as $n \to \infty$. Thus we see from the expression for $\|\dot{c}\|_p^2$ that (iii) is contradicted and so no such sequence of curves exists. ■

The p-boundary thus far appears to reflect the physical situation in the closed Friedmann two-manifold N more reasonably than does the b-boundary. Of course, it is not permissible to conclude from the nondegeneracy in $\tilde{O}^+N$ and the separation of initial and final singularities in $\tilde{\partial}N$ that the same is enjoyed by $\tilde{O}^+M$ and $\tilde{\partial}M$. For, whereas $\tilde{d}_{p_N}(x, y) = 0$ implies $\tilde{d}_{p_M}(h(x), h(y)) = 0$, the converse may be false. However, in the present circumstances it seems unlikely that degeneracy will arise in any of the fibres of $\tilde{O}^+M$ because of the essential interchangeability of the three circular coordinates σ, θ, φ.

3.10. The Topologies of $\tilde{O}^+N$ and $\tilde{\partial}N$; $\bar{N}$ is Hausdorff. Here we use methods similar to those in 3.6 due to Johnson [38]. The extra vertical component in the norm $\| \ \|_p$ compared with $\| \ \|$ has removed the degeneracy of fibres and the part of $\tilde{O}^+N$ over $\psi = 0$ turns out to be a cylinder $S^1 \times \mathbb{R}$. Hence the projection $\tilde{\partial}N$ is a circle, with one point for each $\sigma \in S^1$. We can expect a similar result for the final singularity as well, in the closed Friedmann case.

The vertical component in $\| \ \|_p$ prevents the vertical sequences with constant projection in N from converging to $\tilde{O}^+N$, and so we expect points of $\tilde{\partial}N$ to be Hausdorff separable from points of N. This is indeed the case.

Lemma. The subspace of $\tilde{O}^+N$ over $\psi = 0$ is a cylinder.

Proof. We adapt some definitions used by Johnson [38] for the Schmidt process (see 3.6). Let the *essential p-boundary* of N be $\Pi_{\tilde{O}}(\check{O}^+N)$ where

$$\check{O}^+N = \left\{ \lim_{t \to t_2} c(t) \in \tilde{O}^+N \,\middle|\, c\colon [t_1, t_2) \to O^+N \text{ is inextensible, finite, with } c^1(t) \text{ not bounded away from zero} \right\}$$

We take the definitions of P_0 and c_0 for $\sigma_0 \in S^1$ from 3.6. The technique is to show that $\tilde{O}^+N$ is homeomorphic to $S^1 \times \mathbb{R}$. First consider the map

$$f_0: S^1 \to P_0: \sigma_0 \mapsto \lim_{\psi \to 0} (\psi, \sigma_0, 0)$$

It is surjective by construction. We prove that it is injective. Suppose $\sigma_0, \sigma_1 \in S^1$ are such that $f_0(\sigma_0) = f_0(\sigma_1)$. Then there exists a sequence $\psi_n \to 0$ and a sequence of curves

$$\lambda_n: [0, 1] \to O^+N: t \mapsto (\lambda_n{}^1(t), \lambda_n{}^2(t), \lambda_n(t))$$

such that $\lambda_n(0) = (\psi_n, \sigma_0, 0)$, $\lambda_n(1) = (\psi_n, \sigma_1, 0)$ and satisfying the length condition

$$\lim_{n \to \infty} \int_0^1 \|\dot{\lambda}_n(t)\|_p \, dt = 0$$

If $\sigma_0 \neq \sigma_1$, then we cannot have $\dot{\lambda}_n{}^2 \to 0$. Now, by inspection of the form of $\| \; \|_p$ in 3.9, Lemma 2, we see that our length condition on the λ_n certainly fails if $\dot{\chi}_n$ does not tend to zero. But then we have by positive definiteness

$$\|\dot{\lambda}_n\|_p \geqslant \left| \frac{\dot{R}}{R} \dot{\lambda}_n{}^2 \right| \sim |\dot{\lambda}_n{}^2/\psi_n| \to 0$$

So we may conclude that $\sigma_0 = \sigma_1$ and f_0 is injective.

We can find a similar bijection f_χ for any fixed $\chi \in \mathbb{R}$. Hence we have a bijection

$$f: S^1 \times \mathbb{R} \to \left\{ \lim_{\psi \to 0} (\psi, \sigma, \chi) | (\sigma, \chi) \in S^1 \times \mathbb{R} \right\}$$
$$: (\sigma, \chi) \mapsto \lim_{\psi \to 0} (\psi, \sigma, \chi)$$

We show that this is continuous. Suppose that

$$s: \mathbb{N} \to S^1 \times \mathbb{R}: n \mapsto (\sigma_n, \chi_n)$$

is convergent, in the standard topology, to (σ_0, χ_0). We shall prove that $f \circ s$ is convergent to $f(\sigma_0, \chi_0)$ in $\tilde{O}^+N$.

Let $t_n = |\sigma_n - \sigma_0|^{1/2}$, which by convergence of s is convergent to $0 \in \mathbb{R}$ and which for all large enough n we may suppose lies in $(0, 2\pi)$. We join the point (t_n, σ_0, χ_0) to (t_n, σ_n, χ_n) by a curve $c_n = l_n \cdot k_n$ for each n as follows:

$$k_n: [0, 1] \to O^+N: t \mapsto (t_n, |\sigma_n - \sigma_0|t + \sigma_0, \chi_0)$$
$$l_n: [0, 1] \to O^+N: t \mapsto (t_n, \sigma_n|\chi_n - \chi_0|t + \chi_0)$$

Then

$$\Delta_n = d_p((t_n, \sigma_0, \chi_0), (t_n, \sigma_n, \chi_n)) \leqslant \left| \int_0^1 \|\dot{k}_n(t)\|_p + \|\dot{l}_n(t)\|_p \, dt \right|$$

Evaluating the norms we find, with $R_n = R(t_n)$,

$$\Delta_n \leqslant \left| \int_0^1 \left(R_n(\cosh 2\chi_0)^{1/2} + \frac{\dot{R}_n}{R_n} \right) |\sigma_n - \sigma_0| + 2^{1/2}|\chi_n - \chi_0| \, dt \right|$$

Since

$$R_n \sim t_n{}^{\rho} \quad \text{for some } \rho > 0 \text{ as } n \to \infty \text{ since } t_n \to 0$$

$$\dot{R}_n/R_n \sim 1/t_n = |\sigma_n - \sigma_0|^{-1/2}$$

we may conclude that $\Delta_n \to 0$. Therefore f is continuous because

$$\lim_{n\to\infty} f \circ s(n) = \lim_{t_n \to 0} (t_n, \sigma_0, \chi_0) = f(\sigma_0, \chi_0)$$

Now we show that $f(S^1 \times \mathbb{R}) = \check{\partial}^+N$. Suppose that

$$c\colon [t_1, t_2) \to O^+N\colon t \mapsto (c^1(t), c^2(t), \chi(t))$$

is an inextensible finite curve defining a point of $\check{\partial}^+N$. From the form of the norm and the finiteness condition on c we cannot have $\chi(t) \to \infty$ as $t \to t_2$; likewise we cannot have $c^2(t)$ cycling infinitely around S^1. Hence there must exist limiting coordinates

$$\sigma_0 = \lim_{t\to t_2} c^2(t) \quad \text{and} \quad \chi_0 = \lim_{t \to t_2} \chi(t)$$

Then the same argument as for the continuity of f shows that c defines the same point of $\check{\partial}^+N$ as the radial curve

$$c_0\colon t \mapsto (t, \sigma_0, \chi_0)$$

Hence

$$\lim_{t\to t_2} c(t) = \lim_{t\to 0} (t, \sigma_0, \chi_0) = f(\sigma_0, \chi_0)$$

which establishes $f(S^1 \times \mathbb{R}) = \check{\partial}^+N$.

Finally, f is a homeomorphism because

$$f^{-1}\colon \check{\partial}^+N \to S^1 \times \mathbb{R}\colon \lim_{\psi\to 0} (\psi, \sigma, \chi) \mapsto (\sigma, \chi)$$

exists and is continuous. ∎

Corollary 1. The p-boundary of the Friedmann two-manifold N (nonclosed) is $\Pi_{\check{\partial}}(\check{\partial}^+N) \simeq S^1$. ∎

Corollary 2. For all points $x_0 \in \check{\partial}N$, if $x_1 \in N$ then x_1 can be Hausdorff separated from x_0.

Proof. From the lemma, if $x_0 \in \tilde{\partial}N$ then we can find $(\sigma_0, \chi_0) \in S^1 \times \mathbb{R}$ such that

$$\tilde{x}_0 = \lim_{t\to 0} (t, \sigma_0, \chi_0) \in \Pi_{\check{\partial}}^{\leftarrow}(x_0)$$

Let $x_1\ (\psi_1, \sigma_1) \in N$; then we know that $\psi_1 > 0$.

It is sufficient to show that

$$\tilde{d}_p(\tilde{x}_0, \Pi_{\bar{O}}^{\leftarrow}(x_1)) > 0$$

But this must be so because any curve

$$c\colon [0, 1) \to O^+N\colon t \mapsto (c^1(t), c^2(t), \chi(t))$$

with $c(t) \to \tilde{x}_0$ as $t \to 1$, while $c^1(0) = \psi_1 > 0$ must have $\dot{c}^1$ a nonzero function. Hence $\|\dot{c}\|_p$ will not be the zero function for any curve from the fibre over (ψ_1, σ_1) to the fibre over x_0. The presence of the vertical term due to $\tilde{p}$ in $\|\ \|_p$ prevents our using arbitrarily high horizontal lifts of null curves to achieve zero length in the limit, as was possible in the Schmidt metric (see 3.7.3). ∎

4. THE PRESENT EDGE

In this section we close our account with some notes on the position as it appears after the Eighth International Conference on General Relativity and Gravitation held in August 1977 (see GR8 Abstracts [31]; the contributions on singularities are scheduled to appear in the *Journal of General Relativity and Gravitation*).

4.1. Summary. The theorems of Hawking and Penrose had indicated the likelihood of incomplete geodesics in realistic space–times under very general conditions. For example, they did not depend on particular symmetries, and they used only the attractive character of gravitation, not Einstein's equation. The Schmidt bundle completion had shown considerable promise in topologizing singularities in an elegant way, and Clarke had begun to reveal that generally there are curvature anomalies associated with boundary points. Then Bosshard and Johnson provided details of non-Hausdorff behavior in Friedmann and Schwarzschild space–times. Though such behavior had already been anticipated by Schmidt, it nevertheless raised queries about the physical significance of the b-boundary. Two avenues of development were pursued: modifications to the b-boundary to improve separation properties and classification of singularities in a way compatible with their definition via the b-boundary and its likely modifications. We give some details of these developments in Section 4.2, concentrating on geometrical aspects.

We saw in 3.8 the p-boundary $\tilde{\partial}M$ that is available for space–times having a parallelization $p\colon M \to LM$. It derives from augmentation of the Schmidt metric by an extra vertical component and retains several of the good features of the b-boundary ∂M while avoiding some unwanted identifications. A different modification devised by Clarke (see [31], p. 22) uses a secondary completion to remove certain identifications in the b-boundary. Clarke had expressed the view that the Schmidt metric was basically right but that the process of constructing the boundary from it involves the interior of the space–time instead of concentrating on its edge. To remedy this he used the projective limit topology (see Kowalsky [42], p. 273) on b-boundary closures of sets V_α with compact complements in M. Hence he constructs the essential b-boundary $\partial^\bullet M$, an account of which is given in 4.4 and wherein we find justification for the name. The required separation properties are achieved. Also, the fibre over $x \in \partial^\bullet M$ degenerates to a point if x has *generally singular curvature*; this happens if there is a horizontal curve on which it is impossible to write the curvature tensor R as $R^1 + R^2$ with R^1 bounded and R^2 not exceeding the necessary symmetries. Then if x and y are distinct in $\partial^\bullet M$ with generically singular curvature, these points are Hausdorff separated (see [31], p. 22). Marathe (see [31], p. 243) discussed the alternatives $\tilde{\partial} M$ and $\partial^\bullet M$ and drew attention to his earlier work [48] on paracompactness which leads to a theorem on the metrizability of space–times. We see in 4.3 that he uses a bundle section to construct a topological boundary which is more or less that of Schmidt or Sachs. Marathe observed that Clarke's construction of the projective limit topology for the completion

$$M^\bullet = M^\circ \cup \partial^\bullet M, \quad \text{with } M^\circ \simeq M$$

is similar to his own method of metrizing M, for Clarke's V_α corresponds to the projected interior of certain K_α, of which a component of LM is a countable union (see 4.3). This similarity remains to be exploited, but what we have at present are the proofs (see 4.4) of the following statements.

(i) M° is dense in M.
(ii) Every $x \in \partial^\bullet M$ is Hausdorff separated from any $y \in M^\circ$.
(iii) "Intuitively separate" points are not identified in $\partial^\bullet M$.
(iv) Those points in ∂M without correspondents in $\partial^\bullet M$ lie at the end of a curve partially trapped in some compact set.

Hence we have a satisfactory answer to the difficulties encountered with the b-boundary and a firm position from which cross links can be forged to the classification scheme (4.2) for singularities. We mentioned a sufficient condition for a fibre over $\partial^\bullet M$ to be degenerate ([31], p. 22), but we do not yet know what is necessary. However, it seems unlikely that a manifold structure will be available for the completion in general.

The consensus at GR8, on requiring some bundle section over and above the connection to effect a completion, was that though many reasonable cosmological models admit such sections (see Lee [46] and Geroch [27]) there remained considerable arbitrariness in choosing among the sections. However, it was felt that there was advantage in having such alternatives because of the diversity in singularities and the desirability of having a range of techniques for their investigation. Certainly, in particular situations there may be distinguished sections, derived from matter fields for example. It would be useful to have detailed calculations of different boundaries for comparison; more generally, we need to know when different sections give homeomorphic completions.

A scheme of classifying singularities has been put forward by Ellis and Schmidt [24] (see [31], p. 367) which we outline in 4.2. It supersedes the earlier version in [35] and the more field-theoretic approach of Borzeszkowski and Kasper [6]. King ([31], p. 27) discussed the instability of *whimper* (cf. Ellis and King [23]) or *nonscalar* singularities, where no curvature scalars are badly behaved. Siklos ([31], p. 31) refuted the claim of King and Ellis [40] that whimpers are stable and he commented on their classification. Clarke ([31], p. 22) remarked on the likelihood of boundary points being curvature singularities, and Tipler [63] (see also [31], pp. 32–34) gave bounds on the growth of curvature near singularities. He also extended the rigorous definition of a black hole to space–times that are not asymptotically flat (see [31], p. 221) and gave results on singularities and causality violation (see [31], p. 34). The latter reinforce the general impression [35] that causality violation is insufficient to prevent the formation of singularities in gravitational collapse. Tipler claims that if a closed universe contracts and then reexpands because of causality violation, then the "bounce" must be accompanied by singularities. Furthermore, a space–time with causality violation can be singularity-free only if the causality violation begins in matter-free regions.

More unusual lines of inquiry are the distributional approach to the geometry of singularities by Parker [49] (see also [31], p. 383) and the algebraic computing techniques of Campbell and Wainwright [9] (see also [31], p. 19). The latter gave a list of 14 computable polynomial curvature scalars and studied them near singularities. Scalar polynomial invariants are independent of coordinates; however, there exist nonzero curvature tensors for which all scalar polynomial invariants vanish (cf. [35], p. 260). Thorpe [62] addressed this problem and investigated observer-dependent curvature invariants (see [31], p. 334), which are scalar functions on the unit timelike bundle (see 2.1). These carry direct physical information and throw some light on whether a singular occurrence is tidal (physical forces experienced), matter, or conformal (Weyl tensor misbehaves), for example. It turns out that a timelike curve of bounded acceleration, which could therefore be followed by an observer, runs into a parallel propagated curvature singularity (see [35] and 4.2) if and only if some observer-dependent curvature invariant is unbounded along the curve. We should point out for nonphysicists that the notion of "observer" has a precise mathematical definition, as for example in [53] or [16].

There have been some recent developments of earlier boundary constructions, and we can introduce them by recalling the following. Hawking [33] and Geroch [28] used equivalence classes of incomplete geodesics to construct the geodesic boundary or *g-boundary*. Also, the analysis by Kronheimer and Penrose [43] of causal structures led to the causal or *c-boundary* of Seifert [58] and to the *ideal points* of Geroch, Kronheimer, and Penrose [30]. These constructions were discussed by Hawking and Ellis [35], who concluded that the *b*-boundary was better. Recently, Beem [1] discovered conditions for the stability of causal structures under perturbations of the metric. In particular, if for each compact subset K of space–time $(M, \mathbf{g})$ there is no future inextensible nonspacelike curve that is totally future imprisoned in K, then there is a conformal factor e^{σ} such that every nonspacelike geodesic of $(M, e^{2\sigma}\mathbf{g})$ is complete. Next, Beem [2] compared timelike Cauchy completeness and finite compactness and found them to be equivalent for globally hyperbolic space–times. Moreover, both of these forms of completeness are equivalent to the following: every inextensible future [past] directed geodesic starting in the chronological future [past] of $x \in M$ has points at arbitrarily large distance from x. Woodhouse [64] observed that the study of zero rest mass fields focuses attention on conformal properties of space–time whereas the projective structure is pertinent to massive fields (see II.1.5), but the time-ordering of events is more primitive than either. Hence he devised a *chronological boundary* and compared it with the causal boundary [58]. Subsequently Woodhouse [65] applied Morse theory to geodesics and obtained a link between the causal and curvature structures of stably causal, globally hyperbolic space–times.

It was an early result of Geroch [26] that topology change cannot occur in a compact region without violation of causality in the form of closed timelike curves. We note a few recent additions to this work. Tipler (see [31], p. 34) reports that topology change cannot occur at all in a compact region if the null convergence energy condition (cf. [35], p. 95) holds. Thus topology changes imply singularities. Two related theorems have been proved by C. W. Lee [45]: closed and bounded parts of space cannot change topology in timelike and null geodesically complete space–time; the result is unaffected by allowing the bounding two-manifold (of the part of space in question) to evolve with time. Also, Lee proved a series of theorems on parts of spacelike hypersurfaces enclosed by uniformly convex three-spheres (see [44] for outlines of the proofs and [45] for details). The enclosed part must be compact and simply connected, and the nonenclosed part is noncompact in null and timelike geodesically complete space–times subject to the null convergence energy condition (see [35], p. 95). It is known (K. K. Lee [46], [47]) that no compact space–time can be simply connected and indeed every such space–time has closed timelike curves. Now K. K. Lee

[47] has derived upper bounds on the first and second Betti numbers of space–times having Cauchy surfaces with Abelian fundamental group; hence he obtains a homological classification of space–times with compact Cauchy surfaces.

For lack of space we cannot do justice to the new results on the physics of particular, singular space–times, but we offer some notes on the work reported at GR8. Collins ([31], p. 23) described the behavior of matter variables in some singular cosmologies. Surprisingly, the energy density need not be infinite at a conformal singularity, where only the Weyl tensor has unbounded components in an orthonormal frame. Results for spatially homogeneous cosmologies were reported by Ellis ([31], p. 367) and for Bianchi space–times by Shikin ([31], p. 29). Various singular behaviors were demonstrated by King ([31], pp. 26–27) using stationary cylindrical symmetry in dust-filled space–times. Liang ([31], p. 28) discussed the irrotational collapse of dust, Kegeres ([31], p. 210) included rotation, and Datta ([31], p. 126) described the collapse of perfect fluids.

Finally, mathematicians may regret the nonappearance of applications of catastrophe theory to a seemingly ripe topic. In fact this kite was flown by the organizers of GR8 in their preliminary program, but in the event no papers were forthcoming. The field is open.

4.2. Classification of Singularities. Here we outline the scheme proposed by Ellis and Schmidt [24] (see also [35], p. 367). Suppose that x is a point of the b-boundary ∂M of (M, g). Then the following classes arise.

(i) x is a C^r *regular boundary point* ($r \geqslant 0^-$) if there is an extension (see [35]) of space–time to (M', g') such that the Riemann tensor exists and is C^r for (M, g) and x is an interior point of M'.

Otherwise x is a C^r *singular boundary point*.

(ii) x is a C^r (or C^{r-}) *curvature singularity* ($r \geqslant 0$) if, for some curve c with endpoint x, at least one component of the rth covariant derivative of the curvature tensor, with respect to a parallel propagated orthonormal basis along c, is not continuous (or C^{0-}). Such an x is a singular boundary point in the sense of (i) and was termed a *p·p curvature singularity* in [35].

(iii) If x is a singular boundary point but not in the class of (ii), then it is a C^r (or C^{r-}) *quasi regular singularity* ($r \geqslant 0$). An example is the vertex of a cone. These types were called *locally extensible singularities* in [10] and [23].

(iv) If x is a singular boundary point at the end of a curve c on which some polynomial scalar field constructed from rth covariant derivatives of the curvature tensor is not continuous (or C^{0-}), then x is a C^r (or C^{r-}) *scalar singularity*. These were called *s.p curvature singularities* in [35]. They are necessarily C^r (or C^{r-}) curvature singularities, as in (ii), because the choice of basis field along c is irrelevant, provided that it is C^r.

(v) If x is a curvature singularity but not a C^r (or C^{r-}) scalar singularity, then it is a C^r (or C^{r-}) *nonscalar singularity*. These can coincide with the *intermediate singularities* of [10] and [23].

Ellis and Schmidt gave examples of these types. They also observed that we may wish to distinguish among curvature singularities:

(a) *matter* singularities, when the Ricci tensor causes the problem and

(b) *conformal* singularities, when the Weyl tensor is at fault.

(α) *divergent* singularities, when the relevant components are unbounded and

(β) *oscillatory* singularities, when the relevant components are bounded. Their scheme could easily be adapted to refer to other boundaries for space–time, and though it is intended only for boundary points at "finite distances" from ordinary points, they claim it can be extended to cover those "at infinity."

Clarke [10] has shown that *typically* a point x on the b-boundary will be a curvature singularity, if it is accessible on a timelike or null curve (see also [11] and [31], p. 22). His theorems remain true for a continuous metric tensor with bounded weak derivatives and bounded curvature tensor, but as he says, we are not yet able to conclude that all boundary points are curvature singularities in the Ellis and Schmidt sense. Clarke and Schmidt [13] gave a survey of rigorous results and displayed just what kind of obstruction a space–time singularity is to the extension of the space–time through its boundary and how it can be studied. They distinguished between true singularity theorems and the incompleteness theorems of the Hawking and Penrose type. An example of a singularity theorem of the new type is the following, from Clarke [10] (see also [13]).

> *Theorem.* Let (M, g) be a globally hyperbolic space–time with g of class C^{2-} and suppose we have $x \in \partial M$ such that (i) x is a C^{2-} singular boundary point; (ii) x is the future endpoint of some time-like curve. Then x is a curvature singularity, provided that a certain generality condition (non-D-specialized; see [10], p. 68) holds. ■

So, if the endpoint $x \in \partial M$ of a timelike curve obstructs a C^{2-} extension of curvature through the boundary, then we can expect that on some curve ending at x the curvature components in a parallel propagated frame have no limits. We now have some idea of how this can happen in general because Tipler [63] (see also [31], p. 32) found constraints on the rate of growth of the Ricci curvature near a singularity. He showed that R_{tt} cannot grow faster than t^{-2} near a singularity in any physically realistic space–time, and R_{tt} must grow at least as fast as t^{-1} near a curvature singularity in a conformally flat space–time.

4.3. Completion via a Bundle Section. We studied the p-boundary $\tilde{\partial} M$ for parallelizable space–times in 3.8–3.10 and found that it removed the separation difficulties encountered in the Friedmann sub-space–time by Bosshard and Johnson, while preserving the expected geometry of the completion. More generally, for a given parallelization p, the new metric for $L'M$ is essentially unique, and like Schmidt's $\bar{L}'M$, the completion $\tilde{L}'M$ it provides consists of fibres that are complete, homogeneous spaces. Again holonomy bundles generate the same p-boundary because boundary points of $\tilde{L}'M$ are always accessible on horizontal Cauchy sequences. So most of the good features attributable to the b-boundary persist for $\tilde{\partial} M$. The main disadvantage is in having to decide which parallelization to use, in the absence of some physical guidance such as a matter field and some results on the equivalence of sections in the construction. It is possible, however, that only very close to a singularity will the choice of parallelization affect the completion.

The work of Marathe [48] was previously unknown to those occupied with space–time boundaries. However, Marathe used the Schmidt metric, more or less, to establish the paracompactness of a frame bundle in the process of proving the following result, which also uses a global section.

> *Theorem.* Let LM have structure group G; H is the closed subgroup of G that leaves invariant a given nondegenerate quadratic form, and E is the associated bundle of LM with fibre G/H. Then M is metrizable if E admits a section. (We want M Hausdorff and connected.)

Outline of Proof (see [48]). Consider one component $L'M$. It is locally compact and paracompact, so it can be written (see I.2.6) as a countable union of compact sets K_n such that for all n

$$K_n \subset \operatorname{int} K_{n+1}$$

Each K_n is metrizable and hence so is $\Pi_{L'}(K_n)$.

Now, $\Pi_{L'}$ is open (see II.2.2), and therefore

$$\Pi_{L'}(K_n) \subset \operatorname{int} \Pi_{L'}(K_{n+1})$$

so

$$M = \bigcup_n \Pi_L'(K_n)$$

is also metrizable. ■

> *Corollary.* (i) M is paracompact if E admits a section (see I.2.6).
> (ii) M is paracompact if and only if LM admits a connection.
> (iii) M is paracompact if and only if it admits a pseudo-Riemannian structure. ■

In the context of space–times the candidate for H is of course the Lorentz group, and Geroch [27] had already shown that a space–time metric would imply paracompactness when Marathe observed (see [31], p. 243) that the Cauchy completion of M with the metric from his theorem yields a topological boundary. On the other hand, the connection metrics on LM and its associated bundles yield essentially the Schmidt or Sachs quotient spaces and hence the b-boundary.

4.4. The Essential *b*-Boundary. (This section was written in collaboration with M. J. Slupinski and based on notes supplied by C. J. S. Clarke in amplification of a construction proposed by him; see [31], p. 22.)

4.4.1. Introduction. We suppose that M is a noncompact space–time. For any open subsets V_1, V_2 of M and an isometry

$$i: V_1 \to V_2$$

we shall denote by $\bar{V}_1$, $\bar{V}_2$ the b-boundary (not topological) closures and $\bar{\imath}$ will be the induced map.

We shall use the family

$$J = \{V \subseteq M | M \backslash V \text{ is compact}\} = \{V_\alpha | \alpha \in A\}$$

for some index set A. Later we need the partial order $\leqslant$ defined on A by inclusions among the members of J.

The construction depends on forming a limiting space $M^\bullet$ from a family $\{M_\alpha | \alpha \in A\}$ of topological spaces such that M, V_α, and $\bar{V}_\alpha$ have open homeomorphic images in M_α for each $\alpha \in A$. Then $M^\bullet$ contains a dense subspace $M^0 \simeq M$ and we define the *essential b-boundary* to be

$$\partial^\bullet M = M^\bullet \backslash M^\circ$$

In the sequel, Proposition 1 gives a basis for the topology of $M^\bullet$; Propositions 2 and 3 characterize M^0 and establish its denseness in $M^\bullet$; Propositions 4 and 5 reveal the desired separation properties; Proposition 6 shows that points in ∂M without counterparts in $\partial^\bullet M$ lie at the end of curves partially trapped in some compact set and that points in ∂M arising only from trapped curves have no correspondents in $\partial^\bullet M$.

4.4.2. Preliminaries. For any $V_\alpha \in J$ denote by M'_α the disjoint union of M with $\bar{V}_\alpha$, with topology generated by their disjoint topologies as subbasis. Then M, V_α, and $\bar{V}_\alpha$ have homeomorphic inclusions in M'_α. Now construct the family $\{M_\alpha | a \in A\}$ as follows.

Definition. $M_\alpha = M^\bullet / \sim_\alpha$, $\alpha \in A$, where $\sim_\alpha$ is the equivalence relation induced on M'_α by the inclusion i_α: $V_\alpha \hookrightarrow M$ with $x \sim_\alpha y$ if and only if one of the following holds: (a) $x = y$. (b) $x = i_\alpha(y)$. (c) $i_\alpha(x) = y$. We choose for each M_α the coarsest topology that supports continuity of the projection

$$p_\alpha: M'_\alpha \to M_\alpha: x \mapsto [x]_{\sim_\alpha}$$

It follows that p_α restricts to each of M, V_α, and $\bar{V}_\alpha$ as a homeomorphism, so $p_\alpha V_\alpha$ sits in $p_\alpha \bar{V}_\alpha$ in just the same way that V_α sits in $\bar{V}_\alpha$.

Given any $V_\alpha, V_\beta \in J$ with $V_\alpha \subseteq V_\beta$ (so $\alpha \leqslant \beta$), we can construct a map from M_α to M_β as follows. For $\mu = \alpha$ or β we have

$$\bar{r}_\mu, \quad \text{a homeomorphism of } \bar{V}_\mu \text{ into } M_\mu$$

$$m_\mu, \quad \text{a homeomorphism of } M \text{ into } M_\mu$$

We also have the inclusion maps

$$i_{\alpha\beta}: V_\alpha \hookrightarrow V_\beta \quad \text{and} \quad j_{\alpha\beta}: LV_\alpha \hookrightarrow LV_\beta$$

It is easy to extend these to give

$$\bar{j}_{\alpha\beta}: \bar{L}V_\alpha \to \bar{L}V_\beta: \text{Lim}\,(x_n) \mapsto \text{Lim}\,(j_{\alpha\beta}x_n)$$

$$\bar{i}_{\alpha\beta}: \bar{V}_\alpha \to \bar{V}_\beta: [(x_n)]_G \mapsto [(j_{\alpha\beta}x_n)]_G$$

where

$$[(x_n)]_G = \{\bar{R}_h \,\text{Lim}\,(x_n) | (x_n) \text{ is Cauchy on } LV_\alpha, h \in G\}$$

Since it preserves limits of convergent sequences because $j_{\alpha\beta}$ is metric decreasing, $\bar{j}_{\alpha\beta}$ is continuous. But it need not be injective, for two fibres in $\bar{L}V_\alpha$ may be sent to one fibre in $\bar{L}V_\beta$. Consequently $\bar{i}_{\alpha\beta}$ need not be injective. However, the projections

$$\Pi_{\bar{\alpha}}: \bar{L}V_\alpha \to \bar{V}_\alpha: \text{Lim}\,(x_n) \mapsto [(x_n)]_G$$

$$\Pi_{\bar{\beta}}: \bar{L}V_\beta \to \bar{V}_\beta: \text{Lim}\,(x_n) \mapsto [(x_n)]_G$$

are open, so $\bar{i}_{\alpha\beta}$ is continuous. For if U is open in $\bar{V}_\beta$, then

$$\bar{i}_{\alpha\beta}{}^{\leftarrow} U = \Pi_{\bar{\alpha}} \bar{j}_{\alpha\beta}{}^{\leftarrow}(\Pi_{\bar{\beta}}{}^{\leftarrow} U)$$

is open in $\bar{V}_\alpha$.

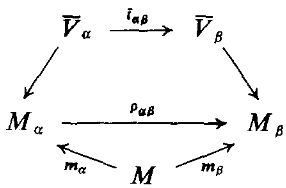
Now we define the required map from M_α to M_β by the following compositions:

$$\rho_{\alpha\beta}: M_\alpha \to M_\beta$$

such that

$$\rho_{\alpha\beta}(x) = \begin{cases} \bar{r}_\beta \circ \bar{i}_{\alpha\beta} \circ \bar{r}_\alpha^{-1}(x) & \text{if } x \in \bar{r}_\alpha(\bar{V}_\alpha) \\ m_\beta \circ m_\alpha^{-1}(x) & \text{if } x \in m_\alpha(M) \end{cases}$$

Though not necessarily injective, the map $\rho_{\alpha\beta}$ is continuous by the continuity of its constituents. We observe, however, that boundary points of V_α may be sent to internal points of V_β.



The Partial Ordering of A. The index A is *decreasingly filtrated* because for all $\alpha, \beta \in A$ we can find $\mu \in A$ with $\mu \leqslant \alpha$ and $\mu \leqslant \beta$. This follows from the fact that $J \cup \varnothing$ is a topology for M, and so

$$V_\alpha, V_\beta \in J \Rightarrow V_\mu = V_\alpha \cap V_\beta \in J$$

We note that $\leqslant$ is only a partial order on A; so it is reflexive, antisymmetric, and transitive; but arbitrary pairs μ, λ from A may satisfy neither $\mu \leqslant \lambda$ nor $\lambda \leqslant \mu$.

4.4.3. The Projective Limit Space $M^\bullet$. We follow Kowalsky [42], p. 273, and define

$$M^\bullet = \lim \text{proj}\, \{M_\alpha, \alpha \in A; \rho_{\alpha\beta}, \alpha \leqslant \beta \in A\}$$

to be the set of all sequences $(x_\alpha)_{\alpha \in A}$ satisfying (a) $x_\alpha \in M_\alpha$ and (b) if $\alpha \leqslant \beta$, then $\rho_{\alpha\beta}(x_\alpha) = x_\beta$. We take as topology for $M^\bullet$ the coarsest that ensures continuity of all projections

$$\chi_\beta \colon M \to M_\beta \colon (x_\alpha)_{\alpha \in A} \mapsto x_\beta, \qquad \beta \in A$$

Evidently, if $\alpha \leqslant \beta$, then

$$\chi_\beta = \rho_{\alpha\beta} \circ \chi_\alpha$$

Lemma. Suppose that U is open in $M^\bullet$ because for some open $V \subseteq M_\beta$ we have $U = \chi_\beta{}^{\leftarrow} V$. Then if $\alpha \leqslant \beta$, we can find open $V' \subseteq M$ with $U = \chi_\alpha{}^{\leftarrow} V'$.

Proof. The identity $\chi_\beta{}^{\leftarrow} = \chi_\alpha{}^{\leftarrow} \circ \rho_{\alpha\beta}^{\leftarrow}$ allows the choice

$$V' = \rho_{\alpha\beta}{}^{\leftarrow} V \quad \blacksquare$$

Proposition 1. The collection

$$T = \{\chi_\alpha{}^{\leftarrow} V \subseteq M^\bullet \mid V \text{ is open in } M_\alpha, \alpha \in A\}$$

is a basis (not just a subbasis) for the topology on $M^\bullet$.

Proof. Consider any finite collaction of open $V_{\alpha_k} \subseteq M_{\alpha_k}$, $k = 1, \ldots, n$, giving rise to the following members of T:

$$U_1 = \chi_{\alpha_1}{}^{\leftarrow} V_{\alpha_1}, \ldots, U_n = \chi_{\alpha_n}{}^{\leftarrow} V_{\alpha_n}$$

We proceed to show that $\bigcap_{k=1}^{n} U_k$ is also a member of T.

Since A is decreasingly filtrated, we can find $\alpha \in A$ with $\alpha \leqslant \alpha_k$ for all $k = 1, \ldots, n$. By the lemma we can find open sets

$$V'_k = \rho_{\alpha\alpha_k}{}^{\leftarrow} V_{\alpha_k} \subseteq M_\alpha, \quad \text{with } U_k = \chi_\alpha{}^{\leftarrow} V'_k$$

hence we see that

$$\bigcap_{k=1}^{n} U_k = \bigcap_{k=1}^{n} \chi_\alpha{}^{\leftarrow} V'_k = \chi_\alpha{}^{\leftarrow}\left(\bigcap_{k=1}^{n} V'_k\right)$$

But the V'_k are open in M_α and therefore so is their intersection; then $\bigcap_{k=1}^{n} U_k \in T$. ■

Definition. $M^\circ = \bigcap_{\alpha \in A} \chi_\alpha{}^{\leftarrow}(m_\alpha M)$.

Proposition 2. M° consists of the constant sequences induced by M and is homeomorphic to M.

Proof. (a) We show that

$$M^\circ = \{(x_\alpha)_{\alpha \in A} \in M^\bullet \mid x_\alpha = m_\alpha(y) \text{ for some fixed } y \in M\}$$

Given $(x_\alpha)_{\alpha \in A} \in M^\circ$, then $x_\alpha \in m_\alpha M$ for all $\alpha \in A$; so suppose that we can find $y, y' \in M$ for which

$$x_\alpha = m_\alpha(y) \quad \text{and} \quad x_\beta = m_\beta(y')$$

As before, we can find $\mu \in A$ such that $\mu \leqslant \alpha$, $\mu \leqslant \beta$, and therefore

$$x_\mu = \rho_{\alpha\mu}(x_\alpha) = \rho_{\beta\mu}(x_\beta)$$

Now, by definition of $\rho_{\alpha\mu}$, $\rho_{\beta\mu}$

$$m_\mu \circ m_\alpha^{-1} \circ m_\alpha(y) = m_\mu \circ m_\beta^{-1} \circ m_\beta(y')$$

So $y = y'$ because m_μ is injective.

(b) We show that for all $\alpha \in A$ the restriction of χ_α to M° is a homeomorphism onto $m_\alpha M$, since we know that $m_\alpha M$ is homeomorphic to M.

Now, $\chi_{\alpha|M^\circ}$ is the restriction of a continuous map and plainly a bijection onto $m_\alpha M$. It remains to prove that it is an open map.

Keep α fixed and consider any $U \in T$, the basis for the topology on $M^\bullet$. Then we have

$V = U \bigcap M^\circ$ is a typical set in the basis for the induced topology on M°

$U = \chi_\beta{}^{\leftarrow} W$ for some $\beta \in A$ and some open set $W \subseteq M_\beta$

Since A is decreasingly filtrated we can find $\mu \in A$ with $\mu \leqslant \alpha$, $\mu \leqslant \beta$, and by the lemma there exists an open set $W' \subseteq M_\mu$ such that

$$U = \chi_\beta{}^{\leftarrow} W' \quad \text{and} \quad V = \chi_\mu{}^{\leftarrow} W' \bigcap M^\circ$$

We arrive at χ_α by

$$\chi_\alpha = \rho_{\mu\alpha} \circ \chi_\mu \quad \text{and} \quad \rho_{\mu\alpha|m_\mu M} = m_\alpha \circ m_\mu^{-1}$$

Hence $\chi_\alpha V$ is open in $m_\alpha M$ because $m_\alpha \circ m_\mu^{-1}$ is a homeomorphism. Open also will be the image by χ_α of any union of sets from the basis for M°. Therefore, $\chi_{\alpha|M^\circ}$ is a homeomorphism. ∎

4.4.4. The Essential b-Boundary $\partial^\bullet M$. We have M°, a homeomorph of M in $M^\bullet$, so we define the boundary

$$\partial^\bullet M = M^\bullet \backslash M^\circ$$

Then by De Morgan

$$\partial^\bullet M = \bigcup_{\alpha \in A} (M^\bullet \backslash \chi_\alpha^{\leftarrow}(m_\alpha M))$$

It follows that $(x_\alpha)_{\alpha\in A} \in \partial^\bullet M$ if and only if (a) $x_\beta \notin m_\beta M$, for some $\beta \in A$ (by definition) or, equivalently, (b) $\exists \beta \in A$ such that for all $\alpha \leqslant \beta$, $x_\alpha \notin m_\alpha M$ (because if β is chosen by (a), then $\alpha \leqslant \beta \Rightarrow \rho_{\alpha\beta} x_\alpha = x_\beta \notin m_\beta M$, so $x_\alpha \notin m_\alpha M$).

Proposition 3. M° is dense in $M^\bullet$.

Proof. Given any $x = (x_\alpha)_{\alpha\in A} \in M^\bullet$ and any open neighborhood N of x, we show that $N \bigcap M^\circ \neq \varnothing$.

We need only consider $x \in \partial^\bullet M$ and N of the form

$$N = \chi_\beta^{\leftarrow} W, \quad \text{for some } \beta \in A$$

with W an open neighborhood of x_β in M_β, because by Proposition 1 such sets form a basis for the topology on M. Therefore we can find some $y \in M$ with $m_\beta(y) \in W$. Hence the sequence

$$(y_\alpha = m_\alpha(y))_{\alpha\in A} \in M^\circ$$

lies in the set $N \bigcap M^\circ$.

Proposition 4. Any $x = (x_\alpha)_{\alpha\in A} \in \partial^\bullet M$ and $y = (y_\alpha)_{\alpha\in A} \in M^\circ$ are Hausdorff separated in $M^\bullet$.

Proof. Since $y \in M^\circ$ we can find $a \in M$ with $\chi_\alpha(y) = m_\alpha(a)$ for all $\alpha \in A$. Also we can choose an open neighborhood V of a in M with $\overline{V}$ compact. Then for some $\mu \in A$ we have

$$V_\mu = M \backslash \overline{V} \in T \quad \text{with } a \notin V_\mu$$

Either x_μ is in ∂V_μ or not.

(i) $x_\mu \notin \partial V_\mu \Rightarrow x_\mu \in m_\mu M$. But $m_\mu(a) \in m_\mu M$ also, and $m_\mu M \simeq M$ is Hausdorff. Hence $m_\mu(a)$ and x_μ can be separated by disjoint open sets W, W' in $m_\mu M$. Therefore x and y are separated by disjoint open sets $\chi_\mu^{\leftarrow} W$, $\chi_\mu^{\leftarrow} W'$ in M.

(ii) $x_\mu \in \partial V_\mu \Rightarrow x_\mu \in \bar{r}_\mu \overline{V}_\mu$. But $\bar{r}_\mu \overline{V}_\mu$ is open in M_μ, and by construction it does not meet $m_\mu V$, which is also open and contains $m_\mu(a)$. Hence the required separation of x and y is effected by the disjoint open sets $\chi_\mu^{\leftarrow}(\bar{r}_\mu \overline{V}_\mu)$ and $\chi_\mu^{\leftarrow}(m_\mu V)$. ∎

We observe that if $V_\mu \in T$, so $M\backslash V_\mu$ is compact; then V_μ may consist of more than one component. However, since M is noncompact, at most one of the components of V_μ is a member of T; for if $V_\mu = V \cup V'$ with $V \cap V' = \varnothing$ and $M\backslash V$ compact, then we cannot have $M\backslash V'$ compact since

$$M = (M\backslash V) \cup (M\backslash V')$$

is noncompact.

We need some formulation of which singularities among the points of $\{\partial V_\alpha | \alpha \in A\}$ are reasonably expected to correspond to distinct points of $\partial^\bullet M$. Then we show that this indeed separates the past and future singularities in closed Friedmann space–time (see 3.7).

Definition. We shall say that $a \in \partial V_\beta$ *corresponds to* $x = (x_\alpha)_{\alpha \in A} \in \partial^\bullet M$ if and only if $x_\beta = a$.

We shall say that $a, b \in \partial V_\beta$ are *intuitively separate* if V_β has two disjoint components V'_β, V''_β with $a \in \partial V'_\beta$ and $b \in \partial V''_\beta$.

> *Proposition 5.* If $a, b \in \partial V_\beta$ are intuitively separate and a, b correspond to $x, y \in \partial^\bullet M$ respectively, then x and y are Hausdorff separated in $M^\bullet$.

Proof. Since V_β is disconnected, the two components V'_β, V''_β whose b-boundaries contain a, b, respectively, are open and closed in V_β. Hence $\partial V'_\beta \cap \partial V''_\beta = \varnothing$ so $\bar{V}'_\beta \cap \bar{V}''_\beta = \varnothing$. But $\bar{V}'_\beta$ and $\bar{V}''_\beta$ are open in M_β. Therefore x and y are separated in M by the disjoint open sets $\chi_\beta{}^{\leftarrow}\bar{V}'_\beta$ and $\chi_\beta{}^{\leftarrow}\bar{V}''_\beta$, respectively. ■

This does separate past from future singular points in M for closed Friedmann space–time, because there we can take a global spacelike slice K that is compact. This leaves the two singular regions in the two disjoint components of $V = M\backslash K$, and $M\backslash V$ is compact by construction.

Finally, we consider points of ∂M that correspond to no points in $\partial^\bullet M$. It turns out that these are closely related to trapped curves (see II.3.10).

Definition. A curve c: $[0, 1) \to M$ is *partially trapped* in a compact set $K \subset M$ if for all $t_k < 1$ there exists some $t > t_k$ with $c(t) \in K$.

> *Proposition 6.* Suppose that $p \in \partial M$ and we denote by $\omega \in A$ the index for $V_\omega = M \in J$, so $\bar{V}_\omega = \bar{M}$. Then we have
>
> (i) $\bar{r}_\omega(p) \notin \chi_\omega(M^\bullet) \Rightarrow \exists c$: $[0, 1) \to M$ ending at p with c partially trapped in some compact K.
>
> (ii) If for some compact K all curves ending at p are partially trapped in K, then $\bar{r}_\omega(p) \notin \chi_\omega(M^\bullet)$.

Proof. (i) If $\bar{r}_\omega(p) \notin \chi_\omega(M^\bullet)$ and no curve ending at p is partially trapped in any compact set, then for all $\mu \in A$ any curve c ending at p remains in V_μ eventually, because $M\backslash V_\mu$ is compact. But such a curve defines a point

$x_\mu \in \bar{r}_\mu \partial V_\mu$ for all $\mu \in A$ and such that $\rho_{\alpha\beta} x_\alpha = x_\beta$ for all $\alpha, \beta \in A$ with $\alpha \leqslant \beta$. However, $\mu \leqslant \omega$ for all $\mu \in A$, so this contradicts $\bar{r}_\omega(p) \notin \chi_\omega(M^\bullet)$.

(ii) If $\bar{r}_\omega(p) \in \chi_\omega(M^\bullet)$, then we can find $(x_\alpha)_{\alpha \in A} \in M^\bullet$ with $x_\omega = \bar{r}_\omega(p)$. For some $\beta \in A$, $V_\beta = M \backslash K$ for the given compact set K. But $x_\beta \in \bar{r}_\beta \partial V_\beta$ and $\rho_{\beta\omega} x_\beta = \bar{r}_\omega(p)$. Hence some curve in V_β ends at p, so it cannot be partially trapped in $K = M \backslash V_\beta$. ■

REFERENCES

1. Beem, J. K. (1976). "Conformal changes and geodesic completeness," *Commun. Math. Phys.*, **49**, 179–186.
2. Beem, J. K. (1976). "Globally hyperbolic space–times which are timelike Cauchy complete," *Gen. Rel. Grav.*, **7**, 339–344.
3. Beem, J. K. (1976). "Some examples of incomplete space–times," *Gen. Rel. Grav.*, **7**, 501–509.
4. Bishop, R. L., and Crittenden, R. J. (1964). *Geometry of Manifolds* (Academic Press, New York).
5. Bishop, R. L., and Goldberg, S. I. (1968). *Tensor Analysis on Manifolds* (MacMillan, New York).
6. Borzeszkowski, H. H. V., and Kasper, U. (1975). "Boundaries of space–times and singularities in general relativity," *Ann. Phys. Leipzig*, **32**, 146–158.
7. Bosshard, B. (1976). "On the *b*-boundary of the closed Friedmann model," *Commun. Math. Phys.*, **46**, 263–268.
8. Brickell, F., and Clark, R. S. (1970). *Differentiable Manifolds* (Van Nostrand, London).
9. Campbell, S. J., and Wainwright, J. (1977). "Algebraic computing and the Newman–Penrose formalism in general relativity," Preprint (Dept. Applied Math., University of Waterloo, Ontario, Canada).
10. Clarke, C. J. S. (1975). "Singularities in globally hyperbolic space–times," *Commun. Math. Phys.*, **41**, 65–78.
11. Clarke, C. J. S. (1976). "Space–time singularities," *Commun. Math. Phys.*, **49**, 17–23.
12. Clarke, C. J. S. (1976). "On the differentiability of space–time," Preprint MPI-PAE/Astro 80 (Max Planck Institute for Physics and Astrophysics, Munich).
13. Clarke, C. J. S., and Schmidt, B. G. (1977). "Singularities: The state of the art," *Gen. Rel. Grav.*, **8**, 129–137.
14. Dieudonné, J. (1960). *Foundations of Modern Analysis* (Academic Press, New York).
15. Dodson, C. T. J. "A new bundle-completion for parallelizable spacetimes," Internal Report IC/77/60 (ICTP, Trieste); presented at GR8 Symposium, Waterloo, Ontario, Canada, August 1977.
16. Dodson, C. T. J., and Poston, T. (1977). *Tensor Geometry* (Pitman, London).
17. Dodson, C. T. J., and Sulley, L. J. (1977). "The *b*-boundary of S^1 with a constant connection," *Lett. Math. Phys.*, **1**, 301–307.
18. Duncan, D. P., and Shepley, L. C. (1974). "A numerical sufficient condition for deciding if a space–time is inextensible," *Nuovo Cimento*, **24B**, 130–134.
19. Duncan, D. P., and Shepley, L. C. (1975). "Boundaries of spacetimes," *J. Math. Phys.*, **16**, 485–492.

20. Eardley, D., and Sachs, R. K. (1973). "Space–times with a future projective infinity," *J. Math. Phys.*, **14**, 209–212.
21. Eells, J. (1974). "Fibre bundles," in *Global Analysis and Its Applications*, vol. I (IAEA, Vienna), pp. 53–82.
22. Ehresmann, C. (1951). "Les connexions infinitésimales dans un espace fibré différentiable," Colloque de Topologie, Bruxelles, 5–8 June 1950, CBRM (Georges Thone, Liege), pp. 29–55.
23. Ellis, G. F. R., and King, A. R. (1974). "Was the big-bang a whimper?" *Commun. Math. Phys.*, **38**, 119–156,
24. Ellis, G. F. R., and Schmidt, B. G. (1976). "Singular space–times," Preprint MPI-PAE/Astro 104 (Max Planck Institute for Physics and Astrophysics, Munich).
25. Friedrich, H. (1974). "Construction and properties of space–time *b*-boundaries," *Gen. Rel. Grav.*, **5**, 681–697.
26. Geroch, R. P. (1967). "Topology in general relativity," *J. Math. Phys.*, **8**, 782–786.
27. Geroch, R. P. (1968). "Spinor structures of spacetimes in general relativity. I," *J. Math. Phys.*, **9**, 1739–1744.
28. Geroch, R. "Local characterization of singularities in general relativity," *J. Math. Phys.*, **9**, 450–465.
29. Geroch, R. P. (1968). "What is a singularity?" *Ann. Phys.*, **48**, 526–540.
30. Geroch, R., Kronheimer, E. H., and Penrose, R. (1972). "Ideal points in space–time," *Proc. R. Soc. Lond. A.*, **327**, 545–567.
31. GR8 *Abstracts of Contributed Papers*, Eighth International Conference on General Relativity and Gravitation, August 7–12, 1977, Department of Applied Mathematics, University of Waterloo, Ontario, Canada.
32. Hájiček, P., and Schmidt, B. G. (1971). "The *b*-boundary of tensor bundles over a space–time," *Commun. Math. Phys.*, **23**, 285–295.
33. Hawking, S. W. (1966). "Singularities and the geometry of space-time," Adams Prize Essay, Cambridge (unpublished).
34. Hawking, S. W. (1971). In *Proceedings of Liverpool Singularities Symposium II*, Lecture Notes in Mathematics 209 (Springer, Berlin), pp. 275–279.
35. Hawking, S. W., and Ellis, G. F. R. (1973). *The Large-Scale Structure of Spacetime* (Cambridge University Press, Cambridge).
36. Hurewicz, W., and Wallman, H. (1948). *Dimension theory* (Princeton University Press, Princeton).
37. Ihrig, E. (1976). "The holonomy group in general relativity and the determination of g_{ij} from T_j^i," *Gen. Rel. Grav.*, **7**, 313–323.
38. Johnson, R. A. (1977). "The bundle boundary in some special cases," *J. Math. Phys.*, **18**, 898–902.
39. Kelley, J. L. (1965). *General Topology* (van Nostrand, Princeton).
40. King, A. R., and Ellis, G. F. R. (1973). "Tilted homogeneous cosmological models," *Commun. Math. Phys.*, **31**, 209–242.
41. Kobayashi, S., and Nomizu, K. (1963). *Foundations of Differential Geometry*, vol. 1 (Interscience, New York).
42. Kowalsky, H-J. (1965). *Topological spaces* (Academic Press, New York).
43. Kronheimer, E. H., and Penrose, R. (1967). "On the structure of causal spaces," *Proc. Camb. Phil. Soc.*, **63**, 481–501.
44. Lee, C. W. (1977). "A restriction on the topology of Cauchy surfaces in general relativity," *Commun. Math. Phys.*, **51**, 157–162.
45. Lee, C. W. (1977). "Implicit restrictions on the topology of space-time," Ph.D. thesis, Dept. Mathematics, University of Lancaster [see also Proc. Roy. Soc. 1978].

46. Lee, K. K. (1973). "Global spinor fields in space–time," *Gen. Rel. Grav.*, **4**, 421–433.
47. Lee, K. K. (1975). "On the fundamental groups of space–times in general relativity," *Gen. Rel. Grav.*, **6**, 239–242.
48. Marathe, K. B. (1972). "A condition for paracompactness of a manifold," *J. Diff. Geom.*, **7**, 571–573.
49. Parker, P. E. (1977). "The Schwarzschild particle," Preprint (Dept. Mathematics, State University of New York at Buffalo).
50. Penrose, R. (1969). "Structure of space–time," in *Battelle Rencontres*, ed. C. M. De Witt and J. A. Wheeler (Benjamin, New York), pp. 121–235.
51. Pontryagin, L. S. (1966). *Topological Groups*, 2nd ed. (Gordon and Breach, New York).
52. Sachs, R. K. (1973). "Spacetime *b*-boundaries," *Commun. Math. Phys.*, **33**, 215–220.
53. Sachs, R. K., and Wu, H. (1976). *General Relativity for Mathematicians* (Springer-Verlag, New York and Berlin).
54. Schmidt, B. G. (1968). "Riemannsche Räume mit mehrfach transitiver Isometriegruppe," Doktorarbeit, University of Hamburg.
55. Schmidt, B. G. (1971). "A new definition of singular points in general relativity," *Gen. Rel. Grav.*, **1**, 269–280.
56. Schmidt, B. G. (1973). "Local completeness of the *b*-boundary," *Commun. Math. Phys.*, **29**, 49–54.
57. Schmidt, B. G. (1974). "A new definition of conformal and projective infinity of spacetimes," *Commun. Math. Phys.*, **36**, 73–90.
58. Seifert, H-J. (1971). "The causal boundary of space–times," *Gen. Rel. Grav.*, **1**, 247–259.
59. Stredder, P. (1975). "Natural differential operators on Riemannian manifolds and representations of the orthogonal and special orthogonal groups," *J. Diff. Geom.*, **10**, 647–660.
60. Steenrod, N. E. (1951). *Topology of Fibre Bundles* (Princeton University Press, Princeton).
61. Thorpe, J. E. (1969). "Curvature and the Petrov canonical forms," *J. Math. Phys.*, **10**, 1–7.
62. Thorpe, J. E. (1977). "Curvature invariants and space–time singularities," *J. Math. Phys.*, **18**, 960–964.
63. Tipler, F. J. (1977). "On the nature of singularities in general relativity," *Phys. Rev. D.*, **15**, 942–945.
64. Woodhouse, N. M. J. (1974). "The completion of a chronological space," *Proc. Camb. Phil. Soc.*, **76**, 531–544.
65. Woodhouse, N. M. J. (1976). "An application of Morse theory to space–time geometry," *Commun. Math. Phys.*, **46**, 135–152.